MISSING INFORMATION AND ASSET ALLOCATION

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Abstract

When the available statistical information is imperfect, it is dangerous to follow standard optimisation procedures to construct an optimal portfolio, which usually leads to a strong concentration of the weights on very few assets. We propose a new way, based on generalised entropies, to ensure a minimal degree of diversification.

One of the major problem in portfolio optimisation and diversification is that the optimal solutions (for example in the risk/return sense of Markowitz [1]) are very often concentrated on a few assets, in contradiction with the very idea of diversification. This feature is considered as unreasonable by most practioners, who feel that it is unwise to discard assets based on the fact that their past performance (or their anticipated performance) is somewhat weaker than those spotted out by the optimisation. Furthermore, due to the instability of the covariance matrix and average return in time, the few assets retained in the allocation tend to evolve with time; a situation which is not satisfactory from an intellectual viewpoint, and furthermore induces large costs. A way out is to impose to the optimisation procedure to remain within preassigned upper and lower bounds for each asset; however, this corresponds, to a large degree, to choosing one's target portfolio, since many assets will have weights equal to the minimum allowed weight.

The aim of the present paper is to introduce a family of 'diversification indicators' which measures how concentrated is the allocation. The optimisation can be performed with a constraint which precludes a too strong 'localisation' of the weights on a few assets. Perhaps more importantly, we relate these indicators to the information content of the portfolio, in the sense of information theory. A strongly concentrated portfolio corresponds to a large information content, while equal weights to all assets corresponds to a minimal information content. We argue that the rationale behind trying to keep a balanced portfolio is that its information content cannot be larger than the information available to the fund manager. This intuitively makes sense: suppose that the average returns and covariances are perfectly known and that one accepts the risk/return framework. The choice of an optimal portfolio on the efficient border is then totally justified, even if the weight of a single asset was 100%. The problem is that one usually has only partial information: past data have a finite length, therefore the determination of statistical parameters is imperfect; predictions of future volatilities and returns are obviously not entirely trustworthy – this is why the optimal portfolio must reflect this lack of information and keep a minimal degree of diversification. This can be implemented using the idea of a 'free-utility' function, in analogy with the free-energy in thermodynamics.

Let p_i, $i = 1, ..., M$ be the weights of asset i in a portfolio, such that $\sum_i p_i = 1$. The problem of characterising how uniform (or how concentrated) these p_i are has been investigated in a very different context, namely that of the phase space of disordered systems [2]. The following quantities were studied:

$$Y_q = \sum_i p_i^q \tag{1}$$

Obviously, for $q = 1$, $Y_1 = 1$. Suppose now that all weights are equal to $1/M$, where M is the total number of assets. Then $Y_2 = 1/M$ and thus goes to zero for large M. On the other hand, if one of the asset has weight p and all the others weight $(1-p)/(M-1)$, one finds $Y_2 = p^2 + O(1/M)$, which thus remains finite for large M. Y_2 (and similarly for all Y_q, $q > 1$) thus distinguishes 'localised' situations, where one, or a few assets gather all the weight, from 'delocalised' configurations, corresponding to a high degree of diversification. Actually, Y_2 is nothing but the *average weight* of each asset. We can thus define an effective number of assets in the portfolio as $M_{\text{eff}} = 1/Y_2$, and look for optimal portfolios with M_{eff} greater or equal than some minimal value M_0. Let us sketch the calculations within a classical Markowitz scheme.

One first checks whether the usual Markowitz solution satisfies the minimum M_{eff} constraint; if not, one then looks for solution constrained to $M_{\text{eff}} = M_0$. Introducing the covariance matrix C_{ij} and the expected return vector r_i, minimisation of the variance of the portfolio for a given average return and for a fixed effective number of assets leads to:

$$\frac{\partial}{\partial p_i}\left[-\sum_{jk} p_j C_{jk} p_k + \lambda \sum_j p_j r_j + \mu \sum_j p_j - \nu \sum_j p_j^2\right] = 0 \tag{2}$$

or, in matrix notation:

$$\boldsymbol{p} = \frac{1}{2}\mathcal{C}^{-1}[\lambda \boldsymbol{r} + \mu \mathbf{1}] \tag{3}$$

with $\mathcal{C}_{ij} = C_{ij} + \nu\delta_{ij}$. The three Lagrange parameters λ, μ, ν are determined as to satisfy the three constraints (sum of weights normalised to 1, fixed average return R and fixed $M_{\text{eff}} = 1/Y_2$). This generates a family of 'sub-efficient border' which is drawn in 1, together with the usual efficient border, which corresponds to $\nu = 0$ (no constraint on Y_2). Technically, Eq. 3 requires, as in the usual case, the inversion of a matrix which is the covariance matrix plus a constant along the diagonal; this constant must be adjusted to satisfy the constraint. Optimisation with a fixed Y_q with $q \neq 2$ is of course possible, although the calculations are somewhat more complex.

The relation with information theory is based on the fact that the indicators Y_q are actually generalised entropies [3]. Indeed, the classical definition of the entropy (or missing information) associated to the weights p_i is [4]:

$$\mathcal{S} = -\sum_i p_i \log p_i \tag{4}$$

which is zero if the portfolio is concentrated on one asset, and maximum for an uniform portfolio ($p_i = 1/M$). It is easily checked that $\mathcal{S}$ can be expressed in terms of the Y_q's as:

$$\mathcal{S} = -\lim_{q \to 1} \frac{Y_q - 1}{q - 1} \tag{5}$$

All the Y_q's play a role similar to the entropy, and can be used to limit the intrinsic informational content associated to the very choice of the weights: the larger the available information, the larger the allowed values of Y_q. It is clear that in the absence of information, the only rational choice is $p_i = 1/N$; while complete information corresponds to $\nu = 0$. It is interesting to note an analogy with thermodynamics: at zero temperature (absence of uncertainty), one must minimise the energy, which is the analogue of (minus) a utility function. At nonzero temperature, one must minimise a 'free-energy' which has both an energetic and entropic contribution. Similarly, one can introduce a general 'free-utility' $\mathcal{F}_q$ which contains an entropic contribution:

$$\mathcal{F}_q = U - \nu \frac{Y_q - 1}{q - 1} \tag{6}$$

where U is a utility function. The above optimisation, Eq. 2, corresponds to $U = \lambda R - \sigma^2$, where σ^2 is the variance of the portfolio. ν plays the role of a 'temperature', which is high when the uncertainty is large. It is obvious that if $\lambda = 0$ (no constraint on the average return), one finds $p_i = 1/M$ in the limit where $\nu \to \infty$. Note that the dimension of ν is the same as that of σ^2. It could be interesting to work in a 'canonical' ensemble where ν, rather than Y_2, is fixed. A natural choice for ν would then be the squared error in the average returns r_i.

In summary, we have argued that optimisation under incomplete information should be performed with constraints on generalised entropies (which are actually diversification indicators), much like in standard thermodynamics. This generates new sub-efficient borders, and allows one to avoid undesired overconcentration of optimal portfolios one very few assets, in accord with general market practice. This idea is not restricted to the Markowitz optimisation scheme, and is useful also when the risk is measured, not as a variance, but as the Value-at-Risk, which in a non-Gaussian world leads to different optimal portfolios [5].

References

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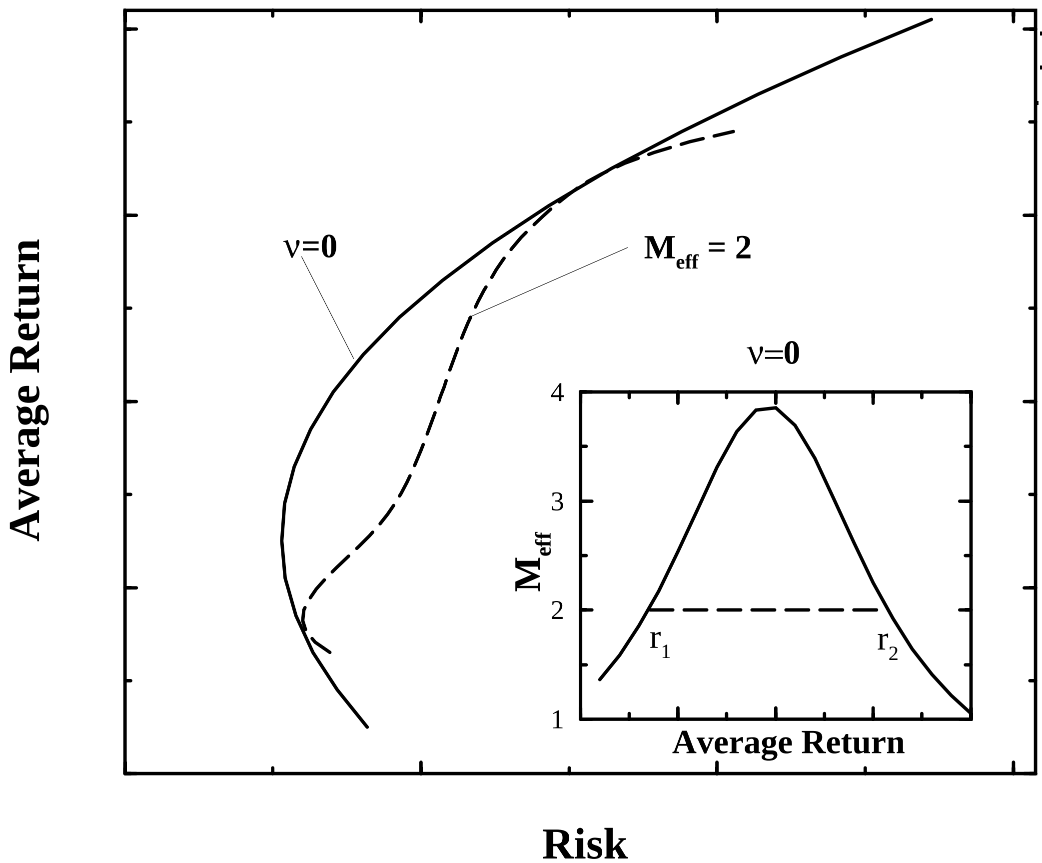

Figure 1: Example of a standard efficient border $\nu = 0$ (thick line) with four risky assets. If one imposes that the effective number of assets is equal to 2, one finds the sub-efficient border drawn in dotted line. The inset shows the effective asset number of the unconstrained optimal portfolio ($\nu = 0$) as a function of average return. The optimal portfolio satisfying $M_{\rm eff} \geq 2$ is therefore given by the standard portfolio for returns between r_1 and r_2 and by the $M_{\rm eff} = 2$ portfolio otherwise.