OPTION PRICING IN THE PRESENCE OF EXTREME FLUCTUATIONS

Jean-Philippe Bouchaud[1,3], Didier Sornette[2,3] and Marc Potters[3]

e de Physique de l'Etat Condensé, CEA-Saclay, 91191 Gif sur Yvette CEDEX, France

toire de Physique de la matière Condensée, Université de Nice-Sophia Antipolis, B.P. 70, Parc Valrose, 06108 Nice CEDEX 2, France.

3) Science et Finance, 109-111 rue V. Hugo, 92 532 Levallois, France.

Abstract

We discuss recent evidence that B. Mandelbrot's proposal to model market luctuations as a Lévy stable process is adequate for short enough time scales, rossing over to a Brownian walk for larger time scales. We show how the easoning of Black and Scholes should be extended to price and hedge opions in the presence of these 'extreme' fluctuations. A comparison between heoretical and experimental option prices is also given.

in the Proceedings of the Newton Institute Session on Mathematical Finance.

I. INTRODUCTION

The efficiency of the statistical tools devised to address the problems of security pricing and portfolio selection strongly depends on the adequacy of the stochastic model chosen to describe the market fluctuations. Historically, the idea that price changes could be modelled as a Brownian motion dates back to Bachelier [1]. This hypothesis, or some of its variants (such as the Geometrical Brownian motion, where the log of the price is a Brownian motion) is at the root of most of the modern results of mathematical finance, with Markowitz portfolio analysis, the Capital Asset Pricing Model (CAPM) and the Black-Scholes formula [2] standing out as paradigms. The reason for success is mainly due to the impressive mathematical and probabilistic apparatus available to deal with Brownian motion problems, in particular Ito's stochastic calculus.

An important justification of the Brownian motion description lies in the Central Limit Theorem (CLT), stating that under rather mild hypothesis, the sum of N elementary random changes is, for large N, a Gaussian variable. In physics or in finance, where these changes occur as time is evolving, the number of elementary changes observed during a time interval t is given by $N = \frac{t}{\tau}$ where τ is an elementary correlation time, below which changes of velocity (for the case of a Brownian particle) or changes of 'trend' (in the case of the stock prices) cannot occur. The use of the CLT to substantiate the use of Gaussian statistics then requires that $t \gg \tau$. In financial markets, τ cannot be smaller than a few seconds which is not that small compared to the relevant time scales (days), in particular when one has to worry about the *tails* of the distribution, corresponding to large shocks (crashes). The Black-Scholes model and many subsequent developments suppose that $\tau \equiv 0$, which enables one to use Ito's stochastic calculus.

Finite values of N thus lead to corrections to Gaussian statistics which one would like to estimate and control. There are however other cases where the Brownian motion model fails, even when $\tau \to 0$. These cases occur either when intertemporal correlation cannot be neglected (the 'fractional Brownian motion' [3] is an example) or when fluctuations are so strong that the second moment of the distribution is infinite – leading to Lévy statistics and stable laws. There is quite a large amount of work making a case for the use of Lévy distributions in finance, starting by B. Mandelbrot's famous 1963 study of coton prices [3,4]. As we shall argue, we believe that the situation is more complicated (as in fact forseen by Mandelbrot himself). In short (but see below), price changes seem to be Lévy-like for short enough time lags and become more and more Brownian as time grows. The *crossover* time T^* between the two regimes depends on the market (currencies, major stock indices, emerging markets..) and is typically a few days for currencies.

The aim of this contribution is first to summarize various important properties of *power-law* tailed distributions, which encompass Lévy stable laws. We then review recent empirical evidence for the 'mixed' behaviour alluded to above, which we shall refer to as the 'truncated' Lévy process. We shall then present a theory for option pricing and hedging in the case of a genuine Lévy process, and finally summarize the theory of option pricing for a general process with a finite variance (but not necessarily Gaussian). In both these cases, perfect hedging is in general impossible, but optimal strategies can be found (analytically or numerically) and the associated residual risk can be estimated – leading to option pricing formulae containing a risk premium.

II. A FEW RESULTS ON POWER-LAW/LÉVY DISTRIBUTIONS

We shall denote in the following the value of the stock at time t as $x(t)$, and δ_t the variation of the stock on a given time interval Δt: $\delta_t = x(t+\Delta t) - x(t)$. The probability density of δ is supposed to be of the form:

$$\rho(\delta) = \frac{\delta_0^\mu}{|\delta|^{1+\mu}} \qquad \text{for} \qquad \delta_0 \ll |\delta| \ll \delta^* \tag{1}$$

where μ is a certain exponent describing how fast the distribution decays to zero, and δ^* an upper cut-off value beyond which ρ decays much faster, say exponentially – see below. These power-law distributions are *scale invariant* (when $\delta^* = \infty$), in the sense that the relative frequency $\frac{\rho(\lambda\delta)}{\rho(\delta)}$ is independant of the chosen scale δ. For $\mu < 1$, the average of δ is of order $E(\delta) \simeq \delta_0^\mu \delta^{*1-\mu}$, and is thus *infinite* when $\delta^* = \infty$. Similarly, when $\mu < 2$, the second moment is of order $E(\delta^2) \simeq \delta_0^\mu \delta^{*\ 2-\mu}$ and also diverges when $\delta^* = \infty$. More generally, only moments $E(\delta^\nu)$ such that $\nu < \mu$ give information on *typical* fluctuations which are of order δ_0.

Where do these power laws come from ? A large number of physical systems actually exhibit truncated power-law distributions of the form given in Eq. (1) [5]. These systems are called 'critical' because they are close to (δ^* large) or right on an instability point ($\delta^* = \infty$). A well known example is the *percolation* problem, where the size of the connected clusters become power-law distributed close to the percolation point [6]. Another trivial example is the probability of first return to the origin after a time t for a (one dimensional) random walk, which decays as $t^{-3/2}$ (or $\mu = 1/2$). More interesting for financial applications are the models exhibiting 'Self Organized Criticality', that is, *spontaneously* evolving towards a critical point [7]. Models of avalanches, earthquakes, crack propagation, etc... have been the subject of intense study in the recent physical literature, and might be relevant to describe bubbles and crashes in the financial markets [8].

Let us now describe a few remarkable properties of power-law distributed variables. Much more precise mathematical statements can of course be given [9]– we deliberately restrict here to a qualitative discussion of the salient features which are useful to our purpose.

• *Extreme values ('Range').* If a set of N of these variables is considered, than the largest value encountered is of order:

$$\delta_{\max} = \max(\delta_1, \delta_2, ..., \delta_N) \propto \delta_0 N^{\frac{1}{\mu}}. \tag{2}$$

Note that $\delta_{\max}$ grows faster for smaller μ's, as expected intuitively.

• *Rank Ordering.* More generally, if one orders these N variables according to their rank, as:

$$y(1) = \delta_{\max}, \quad y(2) = \delta_{\max-1}, \ldots \quad y(N) = \delta_{\min},$$

one obtains the following order of magnitude for $y(n)$:

$$y(n) \propto \delta_0 (\frac{N}{n})^{\frac{1}{\mu}}. \tag{3}$$

This property is actually very useful for empirical characterization of the tail of a distribution (see [13]).

• *Sums* (Total return of a portfolio containing N shares, variation of price over N days, etc.). The order of magnitude of the sum of N independent power-law variables is given by*:

$$S = \sum_{i=1}^{N} \delta_i \propto \begin{cases} N\ E(\delta) & \text{if } \mu > 1 \\ N^{\frac{1}{\mu}} \simeq \delta_{\max}\,! & \text{if } \mu < 1 \end{cases} \tag{4}$$

Note that when $\mu < 1$, the whole sum is of the same order of magnitude as the largest of its terms. This is the most striking aspect of these wildly fluctuating situations which one should keep in mind: few events (rare but important) completely dominate the phenomenon. If $\mu > 1$, on the other hand, the sum is 'democratic': all elementary moves contribute equally to the overall move.

More precisely, when $\mu < 1$, one should rescale S as $u = \frac{S}{N^{\frac{1}{\mu}}}$. The limiting distribution of u for large N is then a symmetric Lévy stable law $L_\mu(u)$ (generalizing the normal law) [9]. An important property of $L_\mu(u)$ is that it decays for large u precisely as the elementary distribution ρ (Eq. (1)): $L_\mu(u) \simeq \frac{\delta_0^\mu}{|u|^{1+\mu}}$. For $1 < \mu < 2$, the mean value $m = E(\delta)$ is finite, and one should consider the rescaled variable $u = \frac{S-mN}{N^{\frac{1}{\mu}}}$. Again, the limiting distribution of u is a Lévy stable law [9]. When $\mu > 2$, one recovers the usual CLT: the rescaled variable $u = \frac{S-mN}{\sqrt{N}}$ becomes Gaussian for large N.

However, it should be emphasized that, for any value of $\mu \in]0, \infty[$, the sum S of individual power-law variables δ_i, all distributed as in Eq. (1), but with possibly *different* 'tail amplitudes' $C_i \equiv \delta_{0i}^\mu$, is also a power-law variable with a tail amplitude C given by

$$C = \sum_{i=1}^{N} C_i. \tag{5}$$

This tail amplitude generalizes the property of the variance, which is additive for independent random variables. The distinction between the cases $\mu < 2$ or $\mu > 2$ lies in the fact that the total weight contained in these power-law tails remain finite in the former case, and decays to zero (as $\frac{N^{1-\frac{\mu}{2}}}{(\log N)^{\frac{\mu}{2}}}$) in the latter case, 'eaten up' by the Gaussian distribution.

• *Truncated power-laws.* If the power-law distribution only extends up to a 'cut-off' or crossover value δ^*, then all the above statements remain qualitatively valid when N is not too large. The simple criterion consists of comparing the order of magnitude of the largest term encountered $\delta_{\max}(N)$ and δ^*. If $\delta_{\max}(N) \ll \delta^*$, or equivalently if $N \ll N^* \equiv (\frac{\delta^*}{\delta_0})^\mu$, then the previous statements apply. Sums of these truncated power-laws will (for $\mu < 2$) first approach a Lévy stable law, and then realize that they are in the attraction basin of the Gaussian for $N > N^*$ [10] – this is graphically represented in Fig. 1.

III. LÉVY DISTRIBUTIONS AND MARKET FLUCTUATIONS

As stated in the introduction, reliable estimates of e.g. option prices require one first to adopt a faithful representation of reality. How faithful is a Lévy process description ? This

---

*The cases $\mu = 1$ or 2 are special: logarithmic corrections need to be included [9] is a much debated issue since Mandelbrot's seminal proposal [4,3,11], with pros and cons which we now summarize.

A point on which everybody agrees is the fact that the kurtosis $\frac{E(\delta^4)}{E(\delta^2)^2}$ is always larger – sometimes much larger – than the Gaussian value of 3. This strong 'leptokurticity' reveals the existence of fat tails, i.e. crashes which would be exceedingly improbable in a Gaussian world. Correspondingly, best fits to Lévy stable laws L_μ systematically favour values of $\mu \simeq 1.6 - 1.8$ rather than the Gaussian value $\mu = 2$. On the other hand, it has also been shown that the main property of stable laws, i.e. to be stable under aggregation, is not well obeyed by the data, and worsens as the time difference Δt increases. Correspondingly, the kurtosis decreases when Δt increases. Furthermore, the concept of an infinite variance seemed so daunting to many (see in particular [12]) that this possibility is often rejected on the basis that it is 'unreasonable' (the same 'common sense' argument was in fact used against these Lévy stable laws in physics for a long time). As we shall show in the next section, optimisation problems such as portfolio selection [13] or option hedging can be well defined even if the underlying process has an infinite variance.

However, we believe that a good representation of market fluctuations is the 'truncated' Lévy process. More precisely, the distribution of price variations at very small time scales τ (of the order of minutes) can be represented as:

$$\rho_\tau(\delta) \simeq \frac{\delta_0^\mu}{\delta^{1+\mu}} \qquad (\delta < \delta^*); \qquad\qquad \simeq \exp -\frac{\delta}{\delta^*} \qquad (\delta > \delta^*). \tag{6}$$

We have obtained evidence for this power-law behaviour followed by an exponential cut-off using different techniques (rank ordered histograms, wavelet analysis) on different type of prices (shares, currencies, etc.) – a detailed account of this study goes beyond the scope of this paper and will be published elsewhere [14]. Interestingly, for many cases studied (although some exceptions exist), the value of μ is remarkably stable, $\mu \simeq 1.6 - 1.8$ (major currencies [14], CAC40 and MATIF [15]). A similar conclusion was recently reached by Mantegna and Stanley [16], who studied the SP Index and found a somewhat smaller value for $\mu = 1.4$, again followed by an exponential cut-off deep in the tails. We should also mention the recent work of Eberlein and Keller [17] which bears many similarities with the present work. In particular, the distribution describing market fluctuations is argued to be *hyperbolic*, which has the same exponential behaviour far in the tails, but a slightly different shape in the center compared to our choice.

As mentioned above, the existence of a cut-off δ^* removes the problem of an infinite variance, but implies the existence of a crossover time $T^* = N^*\tau$ separating a Lévy dominated regime followed by a slow 'creep' towards the Gaussian [10]. This resolves the problem of the 'instability' of the empirical distributions, which becomes manifest for large enough time differences Δt. However, for small Δt, the rescaling of $u = \frac{\delta}{\Delta t^{\frac{1}{\mu}}}$ allows one to "collapse" different histograms on a unique curve, which is indeed very well approximated by a Lévy distribution $L_\mu(u)$: see Fig. 2. The finiteness of δ^* also allows one to rationalize the findings of Olsen et al. [18] who studied the growth of the moments of δ as Δt is increased. For different currencies, they find that

$$E(|\delta|) \propto \Delta t^{0.59} \qquad \text{but} \qquad \sqrt{E(\delta^2)} \propto \Delta t^{0.51}$$

This is precisely what one expects for a truncated Lévy process [†]: it is not difficult to show that in this case:

$$E(|\delta|^\nu) \propto \begin{cases} \delta_0 \ \Delta t^{\frac{\nu}{\mu}} & \text{for } \nu < \mu \\ (\delta^*)^{\nu-\mu} \ \Delta t & \text{for } \nu > \mu \end{cases} . \tag{8}$$

Hence, the results of ref. [18] are in excellent agreement with a truncated Lévy process assumption, with $\mu = \frac{1}{0.59} \simeq 1.7$. It would be very interesting to understand the 'microscopic' origin of such a power-law, and to explain in particular why the value of μ seems to be so 'universal'. The exponential cut-off signals the break down of scale invariance, and is presumably related to external factors, such as allowed bands for currency fluctuations, quotation suspensions, etc.

IV. OPTIONS IN THE PRESENCE OF LARGE FLUCTUATIONS

A. Infinite variance

We now turn to the problem of option pricing and hedging in a 'dangerous' world described by strongly non-Gaussian fluctuations, where crashes are allowed. As mentioned in section 2, the very characteristic of Lévy fluctuations is the dominance of the largest events. Since these events are by definition unpredictable, Lévy markets are necessarily *incomplete* and perfect hedging is impossible. In order to proceed, let us write down the global wealth balance $\Delta W|_0^T$ associated with the writing of a call option:

$$\begin{aligned} \Delta W|_0^T = & \, \mathcal{C}(x_0, x_c, T) \exp(rT) - \max(x(T) - x_c, 0) \\ & + \sum_i \phi(x, t_i) \exp(r(T - t_i)) [\delta_i - r x(t_i) \tau], \end{aligned} \tag{9}$$

where $\mathcal{C}(x_0, x_c, T)$ is the price of the call, T is the maturity, x_c the striking price, $x_0 = x(t = 0)$ and $\phi(x, t)$ the trading strategy. Finally, r the (constant) interest rate and $t_i = i\tau$ is the discrete time. The first term is the gain from pocketing from the buyer the option price at $t = 0$, appreciated to time T. The second term gives the potential loss equal to $-(x(T) - x_c)$ if $x(T) > x_c$ (i.e. if the option is exercised) and zero otherwise. The third term quantifies the effect of the trading and interest between $t = 0$ and $t = T$: the extra variation of wealth W (due to trading) between t and $t + \tau$ is the result of the fluctuations of the stock $(\phi(x, t)\delta_i)$, corrected by the fact that $x\phi(x, t)$ has not benefited from the risk-free interest rate.

We assume that δ_i's are identically distributed power-law variables [‡] ($\delta^* = \infty$) with in general different tail amplitudes δ_0^+ and δ_0^- for positive (resp. negative) variations.

[†]Such a behaviour is however less natural to interpret within the context of hyperbolic distributions [17]

[‡]Note that we assume in the following that price differences and not their logs are power-law distributed.

Since the sum of 'power-law' variables is a power law variable, then the distribution of 'large losses', given by Eq. (9), is a power-law:

$$\rho(\Delta W|_0^T) \simeq_{\Delta W|_0^T \longrightarrow -\infty} \frac{(W_0[\phi(x,t)])^\mu}{|\Delta W|_0^T|^{1+\mu}} \tag{10}$$

with a tail amplitude $W_0[\phi(x,t)]$ which depends on the strategy ϕ. In other words, the probability that the total loss incurred due to trading the option is greater than a certain acceptable loss level $\mathcal{L}$ is given by:

$$P(\Delta W|_0^T < -\mathcal{L}) \simeq_{\mathcal{L}\to\infty} \frac{(W_0[\phi(x,t)])^\mu}{\mu\mathcal{L}^\mu} \tag{11}$$

which serves as a meaningful measure of risk for $\mu < 2$, since in this case the variance of $\Delta W|_0^T$, which we shall use in next section, is infinite. The interesting point about Eq. (11) is that the minimization of risk implies that W_0 should be as small as possible, independently of the value of $\mathcal{L}$. This remark suggests a natural and objective procedure to determine the hedging strategy: the minimisation of large losses selects $\phi^*(x,t)$ such that:

$$\frac{\delta W_0^\mu}{\delta\phi(x,t)}|_{\phi=\phi^*} = 0 \tag{12}$$

where a *functional minimisation* is implied.

Let us study a simple case first, where the trading strategy is trivial $\phi^*(x,t) \equiv \phi^*$ (no rehedging), as can be the case in the presence of very large trading costs. Then $\sum_i \ \phi^* \exp(r(T-t_i))[\delta_i - rx(t_i)\tau] \simeq \phi^*(x(T) - x_0 e^{rT})$ (when $r\tau \ll 1$ [§]). Large losses occur in two cases:

- $x(t)$ drops dramatically: then $\max(x(T) - x_c, 0) = 0$ but there is a loss of $(x(T) - x_0 e^{rT})\phi^*$ due to hold.
- $x(t)$ increases much above x_c: then the option is exercised, inducing a loss of $-(x(T) - x_c)$ with is partially compensated by the hedge. In this case, $\Delta W|_0^T = -(1-\phi^*)x(T) + x_c - \phi^* x_0 e^{rT}$.

The resulting value of W_0 is then easy to compute, using Eq. (5):

$$W_0^\mu = \frac{T}{\tau}\left[\delta_0^{-\ \mu}\phi^{*\ \mu} + \delta_0^{+\ \mu}(1-\phi^*)^\mu\right] \tag{13}$$

The optimal ϕ^* is thus given, for $\mu > 1$, by:

$$\phi^* = \frac{\delta_0^{+\ \zeta}}{\delta_0^{+\ \zeta} + \delta_0^{-\ \zeta}} \qquad \zeta \equiv \frac{\mu}{\mu - 1} \tag{14-a}$$

More generally, one can minimize $P(\Delta W|_0^T < -\mathcal{L})$ for values of $\mathcal{L}$ which are not infinitely large compared to $x_c - x(t)$. One finds in this case that:

$$\phi^*(x,t) = \frac{\delta_0^{+\ \zeta}}{\delta_0^{+\ \zeta} + [\delta_0^- \mathcal{Q}(x,t)]^\zeta} \qquad \mathcal{Q}(x,t) = \left(1 + \frac{x_c - x}{\mathcal{L} + \mathcal{C}[x, x_c, T-t]}\right) \tag{14-b}$$

[§]Note that for $r = 10\%$ per year and $\tau = 1$ day, $r\tau = 2.6\ 10^{-4}$

Once the optimal stategy is known, one can compute the option price by demanding that the *average* gain of the writer of the option must cover part of his potential losses, the order of magnitude of which precisely being $W_0^* \equiv (E(W_0^\mu[\phi^*(x,t)]))^{\frac{1}{\mu}}$. In the simple case where $m = E(\delta) = 0$, $\mathcal{C}(x_0, x_c, T)$ is given by:

$$\mathcal{C}(x_0, x_c, T) = e^{-rT} \int_{x_c}^{\infty} \frac{dy}{\delta_0 T^{\frac{1}{\mu}}} (y - x_c) L_\mu \left(\frac{y - x_0}{\delta_0 T^{\frac{1}{\mu}}} \right) \quad + \beta W_0^*. \tag{15}$$

β is a number of order one, depending on how risk adverse is the writer of the option, fixing a risk premium which be thought of as a bid-ask spread. Note that when $m \neq 0$, the term $+mE(\phi^*(x,t))$ in Eq. (9), representing the average gain (or loss) due to trading, must be subtracted from the option price. In the Black-Scholes limit, this term exactly compensates the difference between $E_{m\neq 0}(\max(x(T) - x_c, 0)$ and $E_{m=0}(\max(x(T) - x_c, 0)$ [20], and one recovers the well known result that $\mathcal{C}(x_0, x_c, T)$ is independant of m.

B. Non Gaussian fluctuations of finite variance

Suppose now that the maturity time scale T which is of interest becomes comparable or larger than the crossover value T^* – imposed by a finite δ^*. In this case, the variance of the wealth variation is a relevant measure of risk (although other ones are possible, such as the fourth moment, etc.., depending on the weight that one wishes to give to the tails). The optimal strategy is then such that the variance of $\Delta W|_0^T$ is minimal [19,22,21]:

$$\frac{\delta E(\Delta W|_0^T[\phi]^2)}{\delta \phi(x,t)} = 0 \tag{16}$$

For a general uncorrelated process (i.e. $E(\delta_i \delta_j) = 0$ for $i \neq j$), the explicit solution of Eq. (16) is relatively easy to write if $m = 0$ and $r = 0$ (the generalisation to other cases is rather more cumbersome):

$$\phi^*(x,t) = \int_{x_c}^{\infty} dx' \langle \delta \rangle_{(x,t) \longrightarrow (x',T)} \frac{(x' - x_c)}{D(x,t)} P(x', T|x, t) \tag{17}$$

where $D(x,t) = E(\delta_i^2)|_{x,t}$ is the 'local volatility' – which may depend on x, t – and $\langle \delta \rangle_{(x,t) \longrightarrow (x',T)}$ is the mean instantaneous increment conditioned to the initial condition (x,t) and a final condition (x', T). The *minimal* residual risk, defined as $\mathcal{R}^* = E(\Delta W|_0^T[\phi^*]^2)$ is in general strictly positive, except for Gaussian fluctuations *in the continuous limit* $\tau = 0$, where one recovers the usual Black-Scholes results ($\mathcal{R}^* = 0$). For $0 < \tau \ll T$, however, the residual risk does not vanish and is given by [19] $\mathcal{R}^* = \frac{1}{2} D\tau \mathcal{P}(1 - \mathcal{P})$, where $\mathcal{P}$ is the probability that the option will be exercized at maturity.

Let us stress that our theory, based on Eq. (16), obviously reproduces the Black-Scholes results in the corresponding limit. Indeed, our starting point, the global balance equation Eq. (9), is nothing but the integrated version of the usual instantaneous balance equation used by Black and Scholes. Note also that approaches related to the minimisation of $E(\Delta W|_0^T[\phi]^2)$ were considered before in the mathematical literature [22], although the optimal strategy, Eq. (17), was not given in explicit form.

Now, in the spirit of the CAPM model, the option price should include a risk premium proportional to the residual risk, and thus be fixed by the equation

$$E(\Delta W|_0^T) = \beta\sqrt{E(\Delta W|_0^T[\phi^*]^2)} \tag{18}$$

where the expectation values are calculated using the empirical distribution $P(x,t|x',t')^{**}$.

The usefulness of Eqs. (17,18) comes from the fact that $P(x,t|x',t')$ can be rather easily reconstructed from empirical data, under the (reasonable) assumption of uncorrelated increments. This has enabled us to calculate numerically the price for real world options – we give an 'experimental' test of our method in Fig.3 [23], on the case of Bund options of short maturities. It is reasonable to assume that on such a liquid market, the risk premium is small (i.e. $\beta = 0$). Fig 3 shows that Eq. (18) with $\beta = 0$ reproduces very well the market prices: the regression gives a slope of 0.9993 ± 0.0009, whereas the Black-Scholes formula (not shown) gives a slope of 1.02 ± 0.002 (and a rather large intercept), which reflects that the latter theory systematically misprices out-of-the-money options.

Furthermore, we can estimate the optimal residual risk, and in particular the dependence of this residual risk on the time lag between rehedging [23]. This is interesting in the presence of transaction costs, where this time lag must realize a trade-off between costly trading and increased risk.

Other interesting cases, such as correlated Gaussian fluctuations (such as the 'Fractional Brownian motion') or option books can be handled with our formalism. We refer the reader to [19,23] for more details.

V. CONCLUSION

Let us summarize the main messages of the present paper:

□ We believe that power-law fluctuations $\mu \sim 1.6 - 1.8$ is a faithful representation of the financial market dynamics *but* only in a finite interval, below a certain cut-off δ^* which depend on the asset. A theory based on Lévy stable laws is thus expected to be most relevant for small enough time scales. For intermediate time scales (weeks), one is right in a *crossover* region, where no simple description is possible, and where formulae such as Eqs. (16,17) are most useful.

□ In the case where the variance is infinite, the correct way to measure the fluctuations and thus the risk is through the 'tail parameter' W_0^μ, which is not very hard to handle analytically thanks to the additivity property Eq. (5). We have shown how option pricing and hedging could be established through minimisation of W_0, yielding formulae for the strategy generalising in an interesting way the Black-Scholes recipe. Finally, the same idea of 'tail chiseling' as a way to control the extreme fluctuations was recently applied to portfolio selection in [13].

**This is, again, similar in spirit to the work of Eberlein and Keller [17], except that the optimal strategy and the residual risk were not considered in their paper.

□ More generally, the precise estimate of the residual risk associated with an option leads to a rational way of fixing a bid-ask spread around the fair price value, which turns out to be a very good estimate of real options, as exemplified in Fig.3.

Acknowledgments: We want to thank many collaborators for sharing with us their skills: J.P. Aguilar, A. Arneodo, E. Aurell, N. Mc Farlane, L. Mikheev, J. Miller, J.F. Muzy, C. Walter and in particular G. Iori. We want to thank Benoit Mandelbrot for many interesting comments and kind encouragments. JPB also wants to thank the organisers and participants of the workshop at the Newton Institute for their warm welcome.

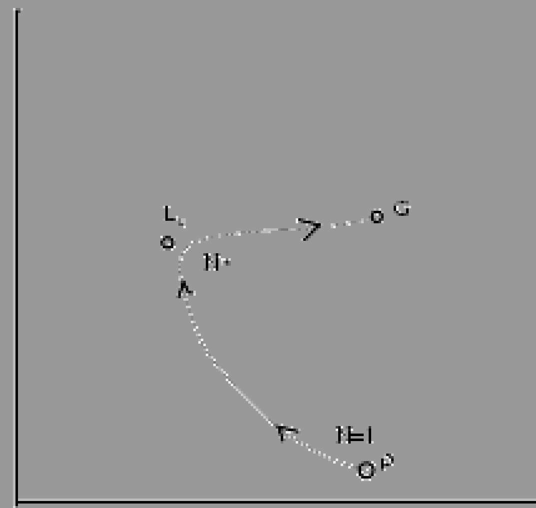


FIG. 1. Graphical representation of the flow of the probability distribution under convolution. When $\delta^* = \infty$, ρ flows towards the fixed point (stable law) L_μ. For finite δ^*, the flow is first directed towards the 'phantom' fixed point L_μ, but after N^* iterations decides that it must flow towards the Gaussian fixed point G.

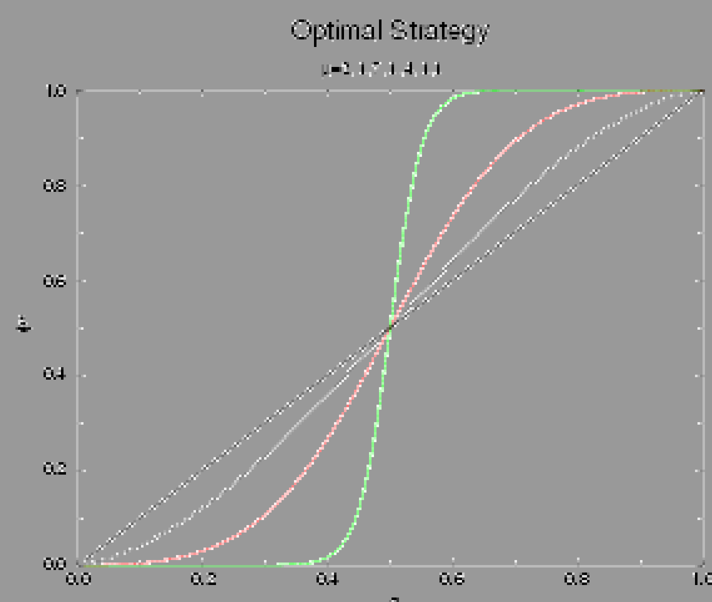
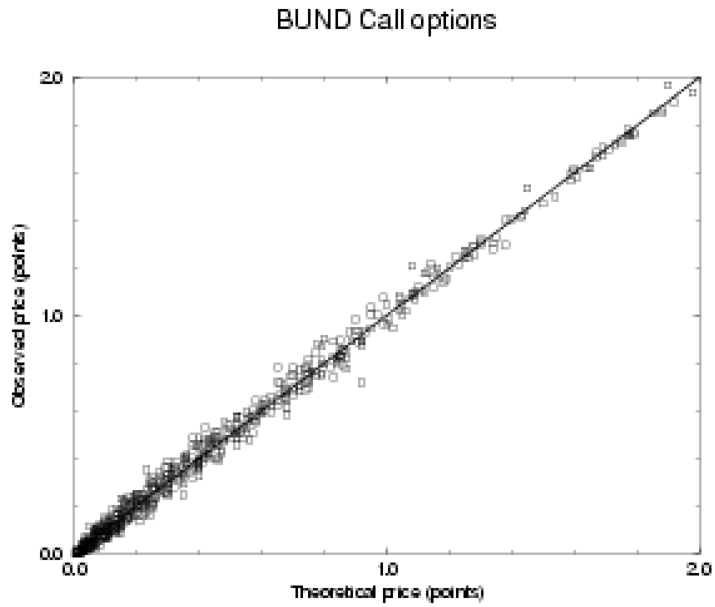


FIG. 2. Rescaled distribution of the price differences, for different time lags, as $\frac{\delta}{\Delta t^{1/\mu}}$, with $\Delta t = a\tau$ and $\mu = 1.7$. The rescaled histogram is very well fitted by a symmetric Lévy distribution L_μ, with the same value of μ (see [14] for details).



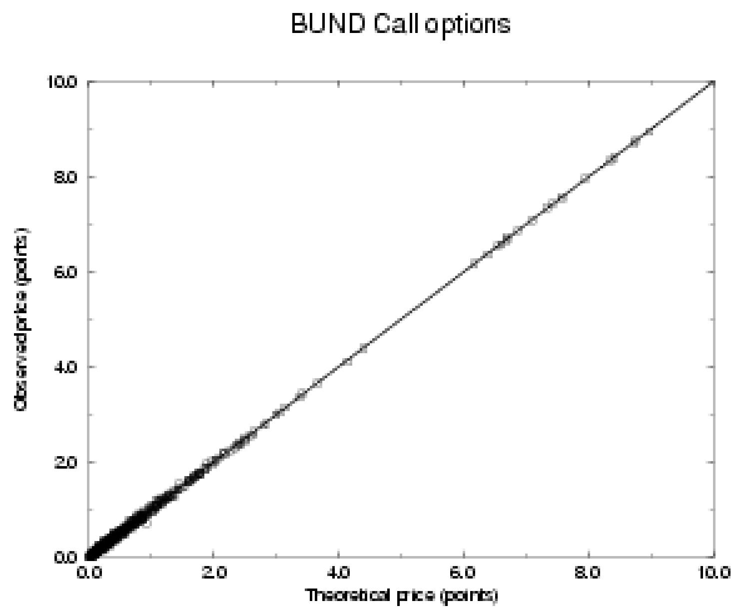


FIG. 3. 'Experimental' prices for Bund call options of different maturities (all less than a month) and strikes between January and June 1995. The data has been extracted from LIFFE's CD-ROM. The coordinate of each point is the theoretical price given by Eq. (18) with $\beta = 0$ on the x axis, and the observed price. $P(x, t|x', t')$ was recontructed using historical data in the period 1992-1994 only. The overall agreement is gratifying, and shows that (i) a truncated Lévy process description is suited to describe (in a first approximation) the whole 'implied volatility' surface; i.e., the way the 'smile' deforms with maturity and (ii) the risk premium is small on very liquid markets. The inset shows the same results on a larger scale.

REFERENCES

[1] L. Bachelier, Theory of Speculation (translation of 1900 French edition), pp.17-78, in Cootner, P.H. editor, The random character of stock market prices (Cambridge, The MIT Press, 1964)

[2] R. C. Merton, 'Continuous-time finance', (Blackwell, Cambridge,1990)

[3] B.B. Mandelbrot, 'The Fractal Geometry of Nature' (Freeman, San Francisco, 1983)

[4] B.B. Mandelbrot, Journal of Business, **36**, 394 (1963), 40, 394 (1967).

[5] 'Lévy flights and related topics in physics', Lecture Notes in Physics vol. 450, G. Zaslavsky, U. Frisch and M. Shlesinger Edts, Springer (1995).

[6] D. Stauffer and A. Aharony, 'Introduction to Percolation theory', second edition (Taylor-Francis, London, 1992).

[7] P. Bak, C. Tang, K. Wiesenfeld, Phys. Rev. Lett., **59**, 381 (1987); D. Sornette, Les phénomènes critiques auto-organisés, Images de la Physique, édition du CNRS, January (1994)

[8] P. Bak, K. Chen, J.A. Scheinkman and M. Woodford, Ricerche Economiche, **47**, 3 (1993); H. Takayasu, H. Miura, T. Hirabayashi and K. Hamada, Physica **A 184**, 127 (1992); T. Hirabayashi, H. Takayasu, H. Miura and K. Hamada, Fractals, **1**, 29 (1993)

[9] B. V. Gnedenko, A.N. Kolmogorov, Limit distributions for sum of independent random variables, Addison Wesley, Reading, MA, (1954); P. Lévy, Théorie de l'addition des variables aléatoires (Gauthier Villars, Paris, 1937-1954); E.J. Gumpel, Statistics of extremes (Columbia University Press, New York, 1960).

[10] R. Mantegna, H.E. Stanley, Phys. Rev. Lett. **73**, 2946 (1994)

[11] for a review, see e.g. Ch. Walter, Bulletin de l'Institut des actuaires français, **349**, 3 (1990) and **350**, 4 (1991) and Thèse, 1994 (unpublished).

[12] P. Cootner, in Cootner, P.H. editor, 'The random character of stock market prices' (Cambridge, The MIT Press, 1964)

[13] J.P. Bouchaud, D. Sornette, Ch. Walter, J.P. Aguilar, preprint (October 1994), submitted to Math. Finance.

[14] A. Arneodo, J.P. Bouchaud, R. Cont, J.F. Muzy, M. Potters, D. Sornette, in cond-mat/9607120, available at http://xxx.lanl.gov, and in preparation.

[15] Ch. Walter, 'Lévy-stable distributions and fractal structure on the Paris market : an empirical examination', Proc. of the first AFIR colloquium (Paris, April 1990), **3**, 241.

[16] R. Mantegna, H.E. Stanley, Nature **376**, 46 (1995)

[17] E. Eberlein, U. Keller, Bernouilli, **1**, 281 (1995)

[18] U.A. Muller, M.M. Dacorogna, R.B Olsen, O.V. Pictet, M. Schwartz, C. Morgenegg, J. Banking and Finance, **14**, 1189 (1990).

[19] J.P. Bouchaud, D. Sornette, J.Phys.I France **4**, 863 (1994),

[20] L. Mikheev, J.Phys.I France **5**, 217 (1995), J.P. Bouchaud, D. Sornette, J.Phys.I France **5**, 219 (1995)

[21] Aurell E. Życzkowski K., "Option pricing & Partial Hedging: Theory of Polish Options" (1995), *Journal of Financial Abstracts: Series D (Working Paper Series)* **3**, January 26 (1996) [abstract], also available as ewp-fin/9601001 at http://econwpa.wustl.edu/wpawelcome.html.

[22] H. Follmer and D. Sondermann, in 'Essays in honor of G. Debreu', W. Hildenbrand, A. Mas-Collel Etaus, North Holland (1986); C.E.R.M.A, C. R. Acad. Sci. Paris, **307**,

625 (1988); H. Follmer, M. Schweitzer, in Applied Stochastic Analysis, eds. M.H.A. Davies and R.J. Elliott (Gordon and Breach Science Publishers, New York, 1990); M. Schweitzer, Stochastic Processes and Their Applications 37, 339-363 (1991).

[23] J.P. Bouchaud, G. Iori, D. Sornette, 'Real world options: smile and residual risk', to appear in Risk Magazine (February 1996), Potters M., Cont R. & Bouchaud J.P., preprint cond-mat/9609172, available at http://xxx.lanl.gov/, submitted to Science.