Explaining the Forward Interest Rate Term Structure

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September 28, 1999

Abstract

We present compelling empirical evidence for a new interpretation of the Forward Rate Curve (FRC) term structure. We find that the average FRC follows a square-root law, with a prefactor related to the spot volatility, suggesting a Value-at-Risk like pricing. We find a striking correlation between the instantaneous FRC and the past spot trend over a certain time horizon. This confirms the idea of an anticipated trend mechanism proposed earlier and provides a natural explanation for the observed shape of the FRC volatility. We find that the one-factor Gaussian Heath-Jarrow-Morton model calibrated to the empirical volatility function fails to adequately describe these features.

1 Introduction

The search for more adequate statistical models of the forward interest rate curve is essential for both risk control purposes and for a better pricing and hedging of interest rate derivative products [3]. A large number of models have been proposed, but it is the Heath-Jarrow-Morton (HJM) model [5] that has become widely accepted as the most appropriate framework for addressing these issues. This model has been the basis for a large amount of research in relation to the pricing and hedging of derivative products. However comparatively little

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has addressed how well this model describes empirical properties of the forward rate curve (FRC).

In a previous paper [1], a series of observations concerning the U.S. FRC in the period 1991-96 were reported, which were in disagreement with predictions of the standard models. These observations motivated a new interpretation of FRC dynamics.

- First, the average shape of the FRC is well fitted by a square-root law as a function of maturity, with a prefactor very close to the spot rate volatility. This strongly suggests that the forward rate curve is calculated by the money lenders using a *Value-at-Risk (VaR) like procedure*, and not, as assumed in standard models, through an averaging procedure. More precisely, since the forward rate $f(t,\theta)$ is the agreed value at time t of what will be the value of the spot rate at time $t+\theta$, a VaR-pricing amounts to writing:
$$\int_{f(t,\theta)}^{\infty} dr' \; P_M(r', t+\theta | r, t) = p, \tag{1.1}$$
where $r(t)$ is the value of the spot rate at time t and P_M is the market implied probability of the future spot rate at time $t+\theta$. The value of p is a constant describing the risk-averseness of money lenders. The risk is that the spot rate at time $t+\theta$, $r(t+\theta)$, turns out be larger than the agreed rate $f(t,\theta)$. This probability is equal to p within the above VaR pricing procedure. If $r(t)$ performs a simple unbiased random walk, then Eq. (1.1) indeed leads to $f(t,\theta) = r(t) + A(p)\sigma_r\sqrt{\theta}$, where σ_r is the spot rate volatility and $A(p)$ is some function of p.

- Second, the volatility of the forward rate is found to be 'humped' around $\theta = 1$ year. This can be interpreted within the above VaR pricing procedure as resulting from a time dependent *anticipated trend*. Within a VaR-like pricing, the FRC is the envelope of the future anticipated evolution of the spot rate. On average, this evolution is unbiased, and the average FRC is a simple square-root. However, at any particular time t, the market actually anticipates a future trend. It was argued in [1] that this trend is determined by the past historical trend of the spot rate itself over a certain time horizon. In other words, the market looks at the past and extrapolates the observed trend in the future. This means that the probability distribution of the spot, $P_M(r', t+\theta|r, t)$, is not centered around r but includes a maturity dependent bias whose magnitude depends on the historical spot trend. However, the market also knows that its estimate of the trend will not persist on the long run. The magnitude of this bias effect is expected to peak for a certain maturity and this can explain the volatility hump.

The aim of this paper is two fold. First we wish to empirically test the new interpretation of the FRC dynamics outlined above. Specifically, we report measurements over several different data-sets, of the shape of the average FRC and the correlation between the instantaneous FRC and the past spot trend over a certain time horizon. We have investigated the empirical behaviour of the FRC of four different currencies (USD, DEM, GBP and AUD), in the period 1987-1999 for the USD and 1994-1999 for the other currencies. Full report of the results can be found in [2]. Here we only present detailed results for the USD 94-99, but we also discuss relevant results obtained with the other data-sets. Second, for USD 94-99, we wish to compare these empirical results with the predictions of the one-factor Gaussian HJM model fitted to the empirical volatility.

2 Empirical Results

Our study is based on data sets of daily prices of futures contracts on 3 month forward interest rates. In the USD case the contract and exchange was the Eurodollar CME-IMM contract. In practice, the futures markets price three months forward rates for *fixed* expiration dates, separated by three month intervals. Identifying three months futures rates to instantaneous forward rates (the difference is not important here), we have available time series on forward rates $f(t, T_i - t)$, where T_i are fixed dates (March, June, September and December of each year), which we have converted into fixed maturity (multiple of three months) forward rates by a simple linear interpolation between the two nearest points such that $T_i - t \leq \theta \leq T_{i+1} - t$. In our notation we will identify $f(t, \theta)$ as the forward rate with fixed maturity θ. This corresponds to the Musiela parameterization. The shortest available maturity is $\theta_{\min} = 3$ months, and we identify $f(t, \theta_{\min})$ to the spot rate $r(t)$. For the USD 94-99 data-set discussed here, we had 38 maturities with the maximum maturity being 9.5 years. We will define the 'partial' spread $s(t, \theta)$, as the difference between the forward rate of maturity θ and the spot rate: $s(t, \theta) = f(t, \theta) - r(t)$. The theoretical time average of $O(t)$ will be denoted as $\langle O(t) \rangle$. We will refer to empirical averages (over a finite data set) as $\langle O(t) \rangle_e$. For infinite datasets the two averages are the same.

First we consider the average FRC, which can be obtained from empirical data by averaging the partial spread $s(t, \theta)$:

$$\langle s(t, \theta) \rangle_e = \langle f(t, \theta) - r(t) \rangle_e. \tag{2.1}$$

In Figure 1 we show the average FRC $\langle s(t, \theta) \rangle_e$, along with the following best fit:

$$\langle s(t, \theta) \rangle_e = a \Big(\sqrt{\theta} - \sqrt{\theta_{\min}} \Big). \tag{2.2}$$

As first noticed in [1], the average curve can be quite satisfactorily fitted by a simple square-root law. The corresponding value of a (in % per $\sqrt{\text{day}}$) is 0.049 which is very close to the daily spot volatility 0.047 (which we shall denote by σ_r). We have found precisely the same qualitative behaviour for our 12 year USD data-set and also for the GBP and AUD. The only exception was the steep DEM average FRC which can be explained by its low average spot level [2]. We have therefore greatly strengthened – with much more empirical data – the proposal of ref. [1] that the FRC is on average fixed by a VaR-like procedure, specified by Eq. (1.1) above.

In figure 2 we show the empirical volatility for the USD, defined as:

$$\sigma(\theta) = \sqrt{\langle \Delta f^2(t, \theta) \rangle_e}, \qquad \sigma(\theta_{\min}) \equiv \sigma_r, \tag{2.3}$$

where $\Delta f(t,\theta)$ denotes the daily increment in the forward rates. We see a strong peak in the volatility at 1 year [3, 4, 1]. For all the data-sets we have studied the volatility shows a steep initial *rise* between the spot rate and 6-9 months forward [2]. We also show the fit of the function:

$$\sigma(\theta) = 0.061 - 0.014 \exp\Big(- 1.55(\theta - \theta_{\min})\Big) + 0.074\ (\theta - \theta_{\min}) \exp\Big(-1.55(\theta - \theta_{\min})\Big). \quad (2.4)$$

It is not *a priori* clear why the FRC volatility should *universally* be strongly increasing for the first few maturities. This is actually in stark contrast to the Vasicek model [3] where the volatility is exponentially decaying with maturity. We will see that this universal feature is naturally explained with the anticipated trend proposal.

We have studied the FRC 'deformation' determined empirically by:

$$y(t,\theta) = f(t,\theta) - r(t) - \langle s(t,\theta)\rangle_e. \quad (2.5)$$

By construction the deformation process vanishes at $\theta_{\min}$ and has zero mean. For the first few maturities we have observed that this quantity is strongly correlated with the past trend in the spot. Therefore, in accordance with the anticipated trend proposal, we consider the following simple one-factor model:

$$f(t,\theta) = r(t) + \langle s(t,\theta)\rangle + \mathcal{R}(\theta) b(t). \quad (2.6)$$

The function $b(t)$ is the 'anticipated trend' which by construction has zero mean. One of the main proposals of [1] was that the anticipated trend reflects the past trend of the spot rate. In other words, the market extrapolates the observed past behaviour of the spot to the nearby future. Here we consider a trend of the form:

$$b(t) = \int_{-\infty}^{t} e^{-\lambda_b (t-t')} dr(t'), \quad (2.7)$$

which corresponds to an exponential cut-off in the past and is equivalent to an Ornstein-Uhlenbeck process for $b(t)$. We have also considered a simple flat window cut-off in [2]. We choose here to calibrate $\mathcal{R}(\theta)$ to the volatility. Neglecting the contribution of all drifts, we find from Eq's. (2.6) and (2.7) that the two are related simply by:

$$\sigma(\theta) = \sigma_r \left[1 + \mathcal{R}(\theta)\right]. \quad (2.8)$$

In accordance with the observed short-end behaviour of the FRC volatility, we require $\mathcal{R}(\theta)$ to be *positive* and strongly increasing for the first few maturities. In our interpretation of the short-end of the FRC, as described quantitatively by Eq's (2.6-8), this universal feature is a consequence of the markets extrapolation of the spot trend into the future.

To determine the parameter λ_b in Eq. (2.7), we propose to measure the following average error:

$$E = \sqrt{\Big\langle \Big(y(t,\theta) - \mathcal{R}(\theta) b(t)\Big)^2 \Big\rangle}. \quad (2.9)$$

To measure E, we must first extract the empirical deformation $y(t,\theta)$ using Eq. (2.5). We then determine $b(t)$ using the empirical spot time series and Eq. (2.7).[1] The error E will have a minimum for some λ_b. This is the time-scale where the deformation and anticipated trend match up best, thereby fixing the values of λ_b. [2] In Figure 3 we plot the error E, against the parameter λ_b^{-1}, used in the simulation of $b(t)$. We consider $\theta = 6$ months which is the first maturity beyond the spot-rate. We see a clear minimum demonstrating a strong correlation between the deformation and anticipated trend. For a flat window model the minimum is even more pronounced [2]. These results indicate the clear presence of a dynamical time-scale around 100 trading days. We have observed that the time-scale obtained is independent of the maturity used [2]. In Figure 4 we plot the empirical deformation against $\mathcal{R}(\theta)b(t)$, where we have set $\lambda_b^{-1} = 100$ trading days. Indeed, we visually confirm a very close correlation. Here we have restricted ourselves to a one-factor model for ease of presentation. In [2] we consider a two and three-factor version of our model where the definition of the deformation now includes the subtraction of a long spread component. In this case we observe improved and very striking correlations that persist even up to 2 years forward of the spot! For the other data-sets the strength of the correlation is not as strong; however the same qualitative features are clearly present.

3 Comparison with HJM

It is important to understand whether the popular HJM framework [5] can capture the empirical properties discussed here. The stationary one-factor Gaussian HJM model is described by:

$$f(t,\theta) = f(t_i, t - t_i + \theta) + \int_{t_i}^{t} ds\ \nu(t+\theta-s) + \int_{t_i}^{t} \sigma(t+\theta-s)dW(s), \tag{3.1}$$

where:

$$\nu(\theta) = \sigma(\theta)\int_0^\theta d\theta'\ \sigma(\theta') - \lambda\sigma(\theta), \tag{3.2}$$

λ is the market price of risk and dW is a Brownian motion. The average FRC is given by:

$$\langle s(t,\theta)\rangle_\tau = f(t_i, \tau+\theta) - f(t_i,\tau) + \int_\tau^{\tau+\theta} du\ \nu(u) - \int_0^\theta du\ \nu(u), \quad \tau = t - t_i, \tag{3.3}$$

which corresponds to an average over a finite time period τ. For comparisons with our empirical average FRC, we can consider τ to be 5 years which was the approximate length of our dataset. There are 3 separate contributions to the average FRC. First is the contribution of the initial FRC. In this case the initial FRC was somewhat steeper than the average FRC. Yet its contribution to the average FRC is still roughly a factor of 3 less than the observed average. We can expect the magnitude of this contribution to decrease with increasing τ.

[1] In this empirical determination of $b(t)$ we actually use detrended spot increments, defined as $d\hat{r}(t) = dr(t) - \langle dr\rangle_e$.

[2] Note that E is also simply the average error between the empirical forward rates and the model forward rates as given by Eq. (2.6).

The second contribution comes from the σ^2 factor in Eq. (3.2). The magnitude of this contribution grows linearly with τ. Yet even for $\tau = 10$ years we find that the size of this term is very small, at least a factor of 10 smaller than the observed average FRC for the early maturities. This term can therefore be neglected. More interesting is the contribution of the market price of risk term. We can show that this contribution is always *negative* for some initial region of the FRC if $\sigma(\theta) > \sigma_r$ for all θ. We found that this condition holds for all the data-sets we studied [2]. This negative contribution has a maximum at $\sigma(\theta) = \sigma(\tau + \theta) \simeq \sigma(\theta_{\max})$. Assuming the volatility is constant for large maturities, we find the market price of risk contribution takes the τ independent form:

$$\langle s(t,\theta) \rangle_\lambda \simeq \lambda \left[\int_0^\theta du\ \sigma(u) - \theta \sigma(\theta_{\max}) \right]. \tag{3.4}$$

In figure 1 we show a plot of Eq. (3.4) where we use the empirical volatility Eq. (2.4) and choose $\lambda = 4.4$ (per $\sqrt{\text{year}}$) which gives a best fit to the average FRC; it is clear that this fit is very bad, in particular compared to the simple square-root fit described above. In the USD case the market price of risk contribution is only negative for the first maturity since the USD has a very strong volatility peak. However for the other data-sets it occurs for much longer maturities or may remain negative for the entire maturity spectrum. Clearly the HJM model completely fails to account for our empirical results regarding the average FRC.

The next question to address is whether the HJM model can explain the striking correlation observed between the deformation and anticipated trend. We do this by calculating Eq. (2.9), where all averages are calculated with respect to the HJM model Eq. (3.1) calibrated to the empirical volatility. As before we have also calibrated $\mathcal{R}(\theta)$ to the empirical volatility via Eq. (2.8). An immediate problem arises because, as we have seen, the HJM average FRC cannot be calibrated to the empirical average FRC. As a result the average deformation will no longer have the required zero mean. We will ignore this problem by defining the deformation as Eq. (2.5) but with the empirical average FRC now replaced by the HJM average FRC. In this case we find the finite τ contributions of Eq. (2.9) are negligible and tend to zero for large τ. The result is plotted in figure 3 where we again consider $\theta = 6$ months. We see that the HJM model fails to adequately account for the strong anticipated trend effect observed here and more strikingly in [2]. This is even after we have, in effect, assumed that the HJM model does describe the correct average FRC. On the other hand, our model is very close in spirit to the strong correlation limit of the 'two-factor' spot rate model of Hull-White [6], which was introduced in an *ad hoc* way to reproduce the volatility hump. Although phrased differently, this model assumes in effect the existence of an anticipated trend following an Ornstein-Uhlenbeck process driven by the spot rate [2]. It would be interesting to understand better the precise relation, if any, between this model and the HJM framework [7].

Our main conclusions are as follows. We confirm with much more data that the average FRC indeed follows a simple square-root law, with a prefactor closely related to the spot volatility. This strengthens the idea of a VaR-like pricing of the FRC proposed in [1]. We also confirm the striking correlation between the instantaneous FRC and the past spot trend over a certain time horizon. This provides a clear empirical confirmation of the anticipated trend mechanism first proposed in [1]. This mechanism provides a natural explanation for the universal qualitative shape of the FRC volatility at the short end of the FRC. This point is particularly important since the short end of the curve is the most liquid part of the curve, corresponding to the largest volume of trading (in particular on derivative markets). Interest rate models have evolved towards including more and more factors to account for the dynamics of the FRC. Yet our study suggests that after the spot, it is the *spot trend* which is the most important model component. Finally, we saw that the one-factor Gaussian HJM model calibrated to the empirical volatility fails to adequately describe the qualitative features discussed here. We presented a simple one-factor version of a more complete model described in [2], which is consistent with the above interpretation.

A natural extension of our work is to adapt the general method for option pricing in a non-Gaussian world detailed in [8] to interest rate derivatives. Work in this direction is in progress.

Acknowledgments:

We thank J. P. Aguilar, P. Cizeau, R. Cont, O. Kwon, L. Laloux, M. Meyer, A. Tordjman and in particular M. Potters for many interesting discussions.

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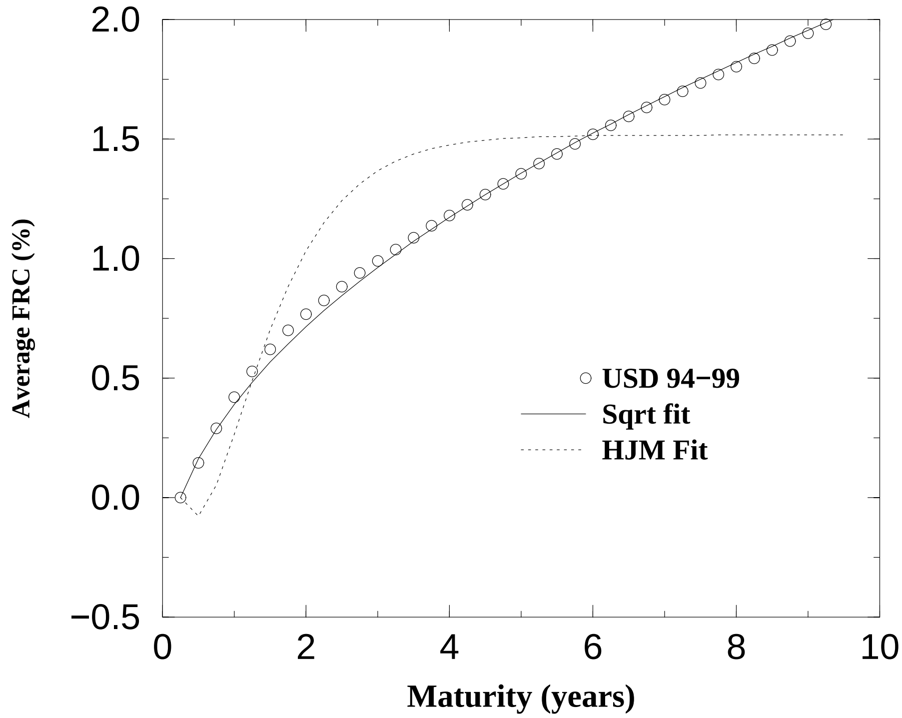

Figure 1: The average FRC for USD 94-99, given empirically by Eq. (2.1), and a best fit to Eq. (2.2). Also shown is the best fit of Eq. (3.4) which is the market price of risk contribution to the average FRC in the HJM framework. This figure demonstrates that the USD average FRC is well fitted by a square root law with a prefactor given approximately by the spot volatility. The HJM framework fails to adequately describe this behaviour.

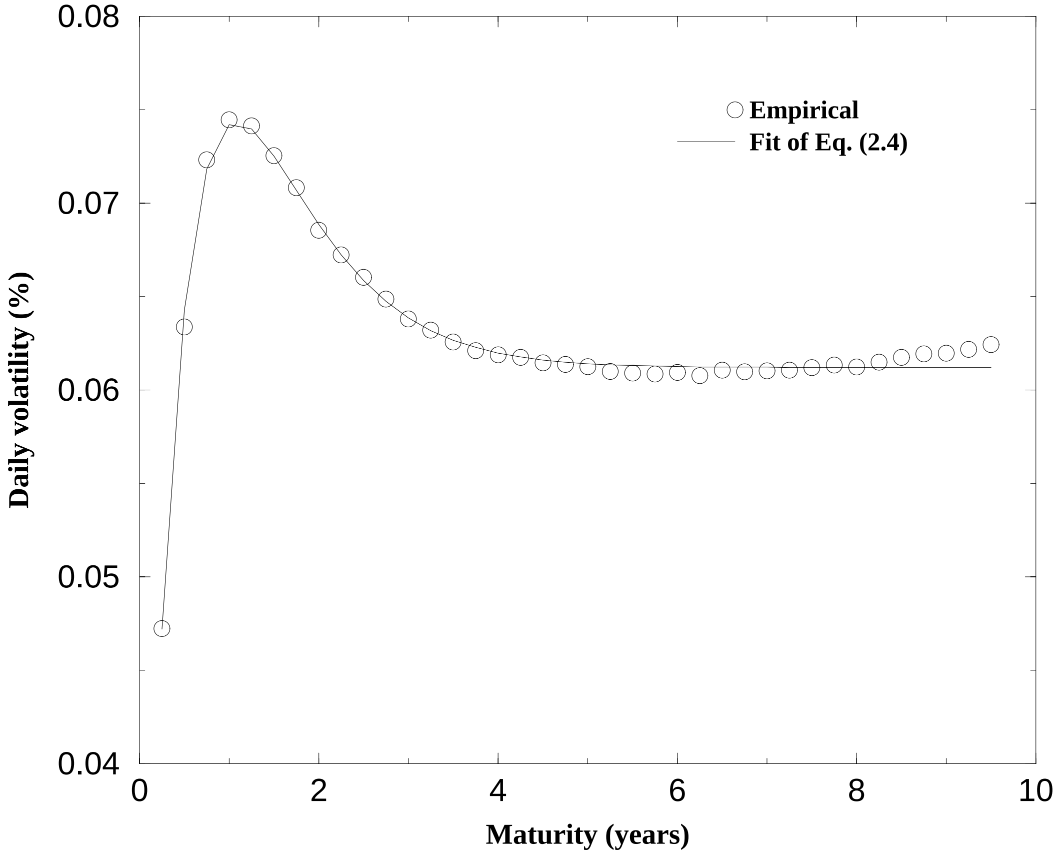

Figure 2: Empirical FRC volatility for USD 94-99 in units of % per square-root day. The empirical volatility is given by Eq. (2.3). Also shown is the fit of Eq. (2.4). The figure demonstrates that the FRC volatility has a strong peak around a maturity of 1 year.

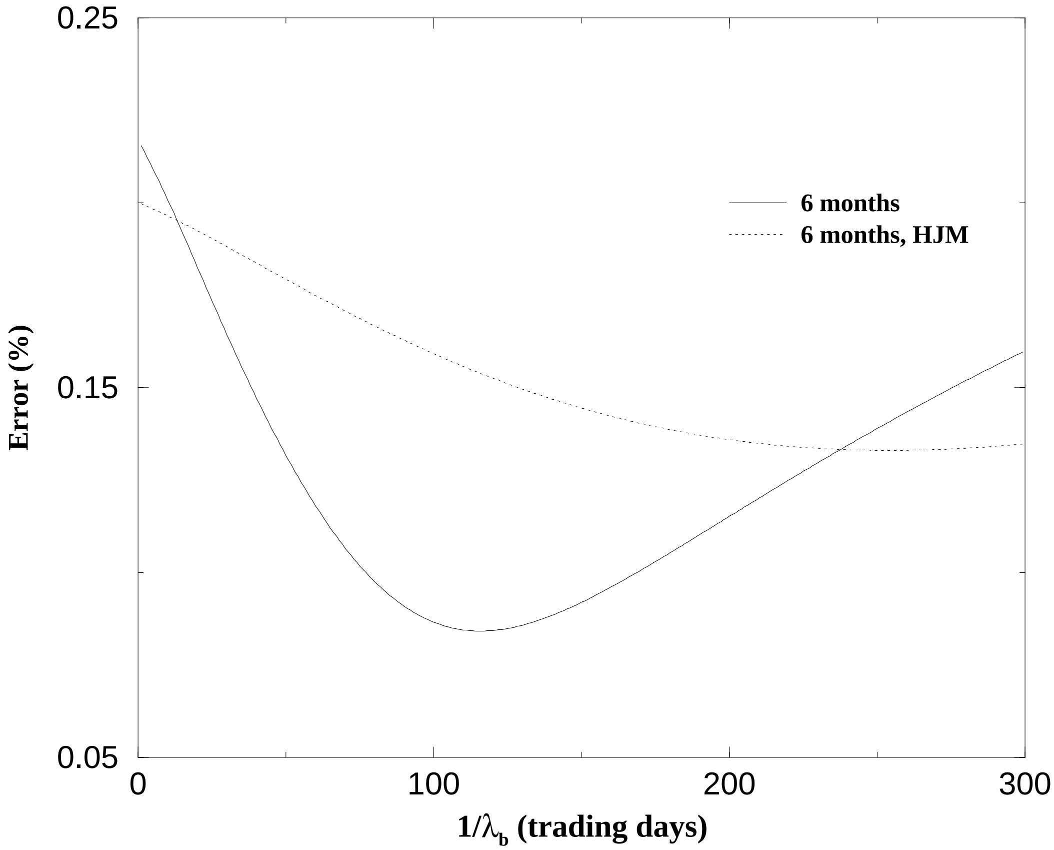

Figure 3: Plot of the error Eq. (2.9), for $\theta = 6$ months, against the parameter λ_b^{-1}, where for the simulation of $b(t)$ we have used Eq. (2.7). Also shown is the error predicted by the one-factor Gaussian HJM model calibrated to the empirical volatility function. This figure demonstrates a strong correlation between the deformation and anticipated trend along with the clear presence of dynamical time-scale in the USD FRC. The HJM model fails to adequately describe these features.

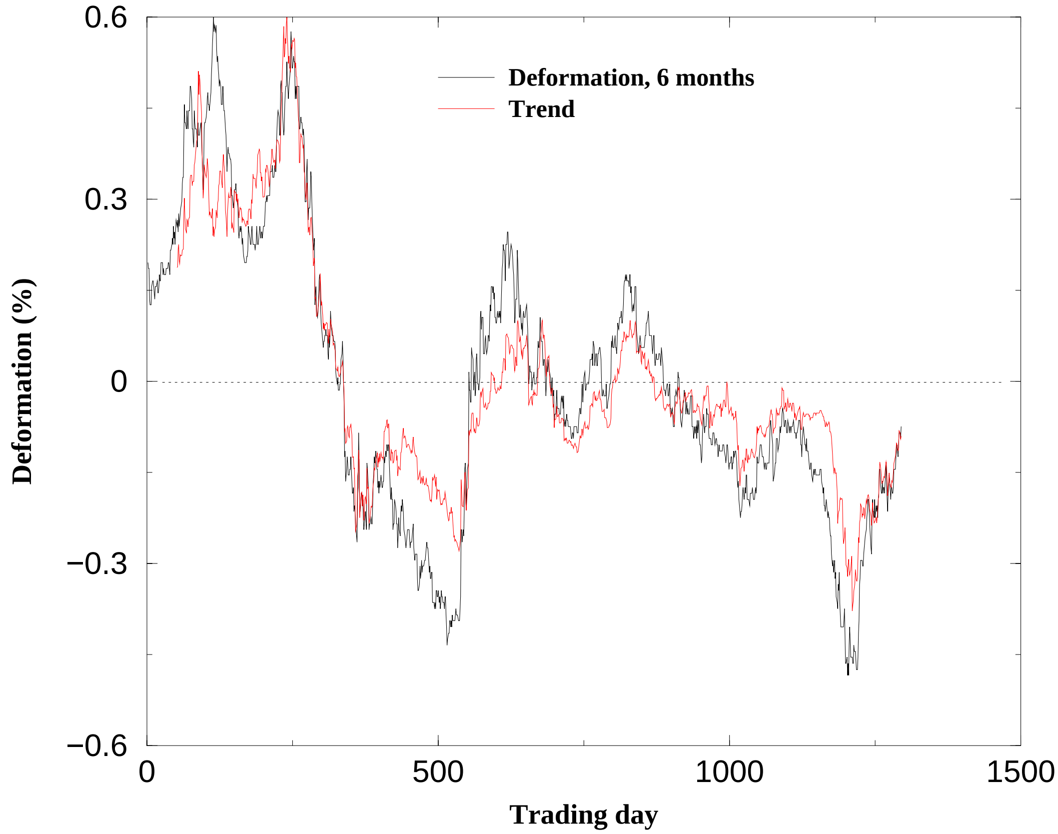

Figure 4: Comparison of the empirical deformation process Eq. (2.5), for $\theta = 6$ months, against the scaled anticipated trend, $\mathcal{R}(\theta)b(t)$. We have calculated $b(t)$ using Eq. (2.7), with $\lambda_b^{-1} = 100$ trading days, while $\mathcal{R}(\theta)$ is calibrated to the FRC volatility. The period covered is 1/1/94 to 18/2/99. This figure demonstrates a strong correlation between the empirical deformation process and the anticipated trend.