Path Dependent Option Pricing: the path integral partial averaging method

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November 1999

Abstract

In this paper I develop a new computational method for pricing path dependent options. Using the path integral representation of the option price, I show that in general it is possible to perform analytically a partial averaging over the underlying risk-neutral diffusion process. This result greatly eases the computational burden placed on the subsequent numerical evaluation. For short-medium term options it leads to a general approximation formula that only requires the evaluation of a one dimensional integral. I illustrate the application of the method to Asian options and occupation time derivatives.

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1 Introduction

Financial derivatives (eg. options and futures) derive their value from an underlying traded financial security, whose price is modeled by some stochastic process. In their most general form, the option payoff is path dependent since it depends on the entire future path traversed by the underlying security. Path dependent options are defined using either discrete or continuous price sampling. For continuous sampling closed from solutions are often available, but in practice most traded path dependent options are discretely sampled. It is known that the application of these closed form solutions leads to substantial pricing errors for discretely sampled options [1, 2, 3]. This feature has necessitated the development of practical and efficient computational methods for the evaluation of path dependent options [4]. Most research has focussed on either partial differential equation, Monte Carlo or tree based methods. In contrast to these approaches, in this paper I will develop an alternative based on the *path integral* formulation of the pricing problem.

For many years, theoretical physicists have been developing and applying the path integral method for calculating expectations similar to those now being encountered in the evaluation of financial derivatives (via the risk-neutral valuation formula). A path integral is an infinite dimensional Riemannian integral as the integral is performed over a set of functions or paths. The path integral method is unique in that it gives a global formulation of the problem in question. This global description provides a powerful tool for deriving analytical approximation and numerical solution schemes that are difficult or impossible to formulate in other ways. Formally, the path integral method is easily extended to multi-dimensional problems. Much of the driving force behind the development of path integral numerical methods in physics has been that, compared to other methods, they show only a slow increase in computational complexity as the dimensionality of the problem increases. Path integral methods were first introduced by Feynman in 1942 as an alternative formulation of quantum physics [5, 6]. They have found wide application in the evaluation of both the real time dynamics and equilibrium statistical mechanics of quantum many-body systems [7, 8, 9]. They are also a popular and natural tool for the analysis of diffusion processes [10, 11, 12], including non-Markovian systems where there is a lack of practical alternatives [14].

The application of path integral methods to financial derivatives was pioneered by Dash who developed a path integral framework for pricing bonds and options within one factor term structure models [15, 16, 17]. More recently, Linetsky [18] was the first to show how path dependent options could be formally priced in a path integral framework. He considered several examples of one and two factor path dependent options. Baaquie [19] has shown how path integral methods can be used to price vanilla options with stochastic volatility. The same author has also recast the popular Heath-Jarrow-Morton model of forward interest rates as a problem in path integration [20]. Similar to Dash, Otto has more recently shown how to use path integration to price bonds and bond options under general short rate models [21]. Bennati *et-al* [22] have focussed on a multi-dimensional path integral formalism for solving general financial problems based on systems of stochastic equations. All these preceding authors focussed on general formalism and exactly solvable models. They have shown that path integrals constitute a natural framework for describing the evaluation of general multi-factor derivatives. Although path integral methods offer an attractive way of obtaining exact solutions, they cannot find exact solutions not available using more standard methods. The greatest promise of path integral methods will be in the development of new numerical and approximation methods for addressing pricing problems where exact solutions are impossible.

Path integral numerical methods involve the evaluation of a multi-dimensional integral, either by deterministic or Monte Carlo methods. A review of deterministic and Monte Carlo methods, developed for physics applications, can be found in Drozdov [11] and Makri [7] respectively. An independent and much more recent development has been the application of path integral numerical methods to financial derivatives. This was pioneered by Eydeland [23] who, using fast Fourier transform methods, has derived a deterministic path integral algorithm for calculating the generating function of a random variable defined as the time integral of a general diffusion process. He pointed to a number of potential financial applications of this algorithm. Chiarella and El-Hassan have applied the method of Eydeland to the pricing of American bond options in the Heath-Jarrow-Morton framework [24]. More recently, Chiarella *et-al* have devised a deterministic path integral algorithm based on Fourier-Hermite series expansions and applied it to the pricing of American options and point barrier options [25, 26]. They report computational times that are a significant improvement over the standard binomial and finite difference methods. In contrast to these deterministic methods, Makivic [28] has initiated research into the path integral Monte Carlo evaluation of financial derivatives. He broadly outlines a computational approach based on the standard Metropolis algorithm. Rosa-clot and Taddei [27] have discussed both a deterministic method and the Monte Carlo approach and applied them to several examples. Some of these previous papers point out that the path integral method is superior to the the traditionally used lattice methods, because the underlying asset price is left continuous rather than being discretized. This has several important advantages. The option price is obtained more accurately since all possible price paths are included in the simulation. Option Greeks are obtained more reliably since the method avoids the need for numerical differentiation. A further advantage of path integral methods is that they can be easily and efficiently extended to evaluate multi-factor financial derivatives.

The application of path integral methods here is fundamentally different from all the previous cited works. We use the path integral representation of the option price to show that rather generally, it is possible to perform analytically a *partial averaging* over the underlying risk-neutral diffusion process. This key result will greatly reduce the computational burden placed on the subsequent numerical evaluation. The application of this method is inspired by a technique first developed for computation problems in chemical physics [30, 31]. Conceptually, the partial averaging corresponds to averaging over the high frequency fluctuations of the risk-neutral diffusion process. For short-medium term options it leads to a general approximation formula that only requires the evaluation of a one dimensional numerical integral. Longer term options can be evaluated deterministically or most generally by standard Monte Carlo methods. In this case, the partial averaging method will greatly reduce the required dimension of the Monte Carlo simulation.

The outline of this paper is as follows. In section 2 we develop a formal path integral representation for the evaluation of rather general path dependent options. In section 3 we show how the previous path integral can be numerically evaluated after performing analytically a partial averaging over the underlying risk-neutral diffusion process. In section 4 some examples are presented. For clarity of presentation and completeness, the path integral methods used in this paper are developed from first principles in the three appendices.

2 Path dependent option theory

In this section we will present a formal path integral framework general enough to price a wide range of path dependent options. We will assume the standard geometric Brownian motion (GBM) model of the asset price process. More general diffusion processes present no special difficulties. In this case the risk-neutral price process is given by the Ito stochastic differential equation

$$dS_t = (r-q)S_t dt + \sigma S_t dW_t, \tag{2.1}$$

where r is the interest rate and q is the continuous dividend yield. Using Ito's lemma we can show

$$dx_t = \mu dt + \sigma dW_t, \tag{2.2}$$

where

$$x_t = \ln S_t, \quad \mu = r - q - \frac{1}{2}\sigma^2. \tag{2.3}$$

Consider a path dependent option with price C_F at expiry u given by

$$C_F(S_u, \mathcal{I}, u) = F(S_u, \mathcal{I}) = F(e^{x_u}, \mathcal{I}), \tag{2.4}$$

where F is the option payoff function which depends on some path dependent random variable $\mathcal{I}$. In this paper we will assume that $\mathcal{I}$ can be written as

$$\mathcal{I} = \int_t^u ds\, w(s) f(x_s, s), \tag{2.5}$$

which is a time integral over an arbitrary function of the risk-neutral diffusion process (2.2). For continuous sampling $w(s) = 1$, but for discrete sampling (which is more realistic in practice)

$$w(s) = \sum_i w_i \delta(s - s_i), \tag{2.6}$$

where w_i are the sampling weights and s_i are the sampling times. The above definition of $\mathcal{I}$ was used before by Wilmott *et-al* [29] whose focus was on partial differential equation methods. It was shown to be general enough to include Asian, barrier and lookback options which are 3 qualitatively different path dependent options. We will present examples in section 4.

In a risk-neutral framework, the option price at inception time t is given by

$$C_F(S_t, t) = e^{-rT} E_{x_t}[F(e^{x_u}, \mathcal{I})], \tag{2.7}$$

where $T = u - t$ and the expectation is with respect to the transformed risk-neutral price process (2.2) conditioned on the initial value x_t. The price at inception can then be expressed as

$$C_F(S_t, t) = e^{-rT} \int_{-\infty}^{\infty} dx_u \int_{-\infty}^{\infty} d\mathcal{I}\, P(x_u, \mathcal{I}|x_t) F(e^{x_u}, \mathcal{I}), \tag{2.8}$$

where $P(x_u, \mathcal{I}|x_t)$ is the joint probability density function (PDF) of x_u and the path dependent random variable $\mathcal{I}$. In appendix A we show for a general diffusion process how the joint PDF can be formally computed as a path integral. For the special case of GBM, we show in appendix A that the joint PDF is given by

$$P(x_u, \mathcal{I}|x_t) = \frac{1}{2\pi} \exp\left[\frac{\mu x_u}{\sigma^2} - \frac{\mu x_t}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right] \int_{-\infty}^{\infty} dk e^{-ik\mathcal{I}} K(x_u, x_t; T), \tag{2.9}$$

where μ is the constant defined in (2.3) and K, which we refer to as the propagator is defined by

$$K(x_u, x_t; T) = \int_{x_t}^{x_u} \mathcal{D}x_s \exp\left[-\frac{1}{2\sigma^2} \int_t^u ds \; \left(\dot{x}_s^2 + V(x_s, s)\right)\right]. \tag{2.10}$$

In (2.10), the integration measure $\mathcal{D}x_s$ denotes a *path integral* which is defined precisely in appendix A. It describes an infinite dimensional integral over all paths connecting x_u at the expiry time and x_t at the initial time. We refer to the function V in (2.10) as the potential function. For the GBM model it has the form

$$V(x_s, s) = -2ik\sigma^2\, w(s)\, f(x_s, s). \tag{2.11}$$

We see that in this case the potential is imaginary with a functional form determined by the path dependent random variable (2.5). We show in appendix A that for more general risk-neutral diffusion processes the potential function is complex.

The propagator (2.10), up to a boundary term, is the characteristic function of the joint PDF. In physics, the path integral (2.10) is equivalent to the path integral representation for the equilibrium quantum statistical density matrix of a particle in a complex potential $V(x)$ [5]. In this case temperature replaces the role of time in (2.10). Under an imaginary time transformation, (2.10) becomes equivalent to that which gives the quantum mechanical propagator of a one dimensional quantum particle in a complex potential $V(x)$. These identifications with standard problems in theoretical physics are of great value because we can then use the methods and results of theoretical physics for performing these path integrals. Tables of known exact path integrals of the form (2.10), for various potential functions, are listed by Grosche [6]. Such tables along with the formulation provided here provide an easy way to obtain exact solutions to path dependent option prices.

2.1 Seasoned path dependent options

In this paper we consider the option at its inception time. More generally, the option price at time t' ($t < t' < u$) for a seasoned path dependent option is given by

$$C_F(S_{t'}, \mathcal{I}_t^{t'}, t') = e^{-r(u-t')} E_{x_{t'}}[F(e^{x_u}, \mathcal{I}_t^{t'} + \mathcal{I}_{t'}^u)], \tag{2.12}$$

where we use the more detailed notation

$$\mathcal{I}_t^u = \int_t^u ds\, w(s) f(x_s, s). \tag{2.13}$$

We then find

$$C_F(S_{t'}, \mathcal{I}_t^{t'}, t') = e^{-r(u-t')} \int_{-\infty}^{\infty} dx_u \int_{-\infty}^{\infty} d\mathcal{I}_{t'}^u \, P(x_u, \mathcal{I}_{t'}^u | x_{t'}) F(e^{x_u}, \mathcal{I}_t^{t'} + \mathcal{I}_{t'}^u). \tag{2.14}$$

We see that this case only affects the payoff function in a simple way and the joint PDF we need to find is the same problem as before. Therefore all the final results can be trivially extended to seasoned options.

3 Partial Averaging

In the previous section we showed that for the GBM model, equations (2.5) and (2.9-11) define a formal path integral representation for the joint PDF. The option price is then obtained from this joint PDF via (2.8). In this section we use the previous path integral formulation to show that it is possible to perform analytically a partial averaging [30] over the underlying risk-neutral diffusion process. The option price can then be more efficiently evaluated by numerical methods.

First we must discretize in time (2.2), by defining the discrete time $s_n = n\varepsilon + t$, where $n = 0, 1, ..., N$ and $\varepsilon = T/N$ with $T = u - t$. The propagator (2.10) can then be decomposed as

$$K(x_u, x_t; T) = \int_{-\infty}^{\infty} dx_{N-1} ... dx_1 \prod_{n=1}^{N} K(x_n, x_{n-1}; \varepsilon) \tag{3.1}$$

where $x_N \equiv x_u, \; x_0 \equiv x_t$ and a general form for the short-time propagator is, as shown in appendix C,

$$K(x_n, x_{n-1}; \varepsilon) = \left(\frac{1}{2\pi\sigma^2\varepsilon}\right)^{1/2} \exp\left[-\frac{(x_n - x_{n-1})^2}{2\sigma^2\varepsilon} - \gamma(x_n, x_{n-1}; \varepsilon)\right], \tag{3.2}$$

where γ will be defined below. Substituting (3.2) into (3.1) we find that the propagator becomes

$$K(x_u, x_t; T) = \left(\frac{1}{2\pi\sigma^2\varepsilon}\right)^{\frac{N}{2}} \int_{-\infty}^{\infty} dx_{N-1} ... dx_1 \exp\left[-\sum_{n=1}^{N} \frac{(x_n - x_{n-1})^2}{2\sigma^2\varepsilon} - \sum_{n=1}^{N} \gamma(x_n, x_{n-1}; \varepsilon)\right]. \tag{3.3}$$

As $\varepsilon \to 0$, its possible to show that

$$\gamma(x_n, x_{n-1}; \varepsilon) = \varepsilon \frac{(V(x_n, s_n) + V(x_{n-1}, s_{n-1}))}{4\sigma^2} \tag{3.4}$$

yields the correct short-time propagator when substituted into (3.2).

We will refer to equation (3.2), with (3.4), as the primitive short-time propagator. It is the standard short-time propagator used in the numerical evaluation of path integrals. A key feature of this propagator is that it is *not* correct to first order in ε. It is in fact only valid as $\varepsilon \to 0$. Clearly, the dimension of the integral in (3.1) could be made much smaller by searching for short-time propagators accurate over larger time-steps. This observation has motivated the search for improved short-time propagators for use in path integral calculations for physics applications [7, 13, 31]. In appendix C, we show that its possible in general to write

$$\gamma(x_n, x_{n-1}; \varepsilon) = -\sum_{m=1}^{\infty} \frac{1}{m!} \left(-\frac{\varepsilon}{2\sigma^2}\right)^m C_m(x_n, x_{n-1}; \varepsilon), \tag{3.5}$$

where C_m describes a cumulant structure with each cumulant of order $(\sigma^2\varepsilon)^{m-1}$. The key result is, we can obtain a short-time propagator formally correct to *second* order in ε, by truncating all cumulants beyond the first. This truncation corresponds to retaining an averaging over only the high frequency fluctuations of the risk-neutral diffusion process; i.e. a partial averaging. The improved short-time propagator will lead to a much more efficient numerical evaluation compared to that obtained by using the primitive short-time propagator. In appendix C, we calculate the first 2 cumulants exactly for a general potential and derive an expansion of the propagator to third order in T (a result only valid for smooth potentials).

Using the results from appendix C, with the GBM potential (2.11), we find that

$$\gamma(x_n, x_{n-1}; \varepsilon) \simeq -i\varepsilon k\alpha(x_n, x_{n-1}; \varepsilon) + \frac{\varepsilon^2 k^2}{2}\beta(x_n, x_{n-1}; \varepsilon) + o(\sigma^4\varepsilon^5), \tag{3.6}$$

where

$$\alpha(x_n, x_{n-1}; \varepsilon) = \int_0^1 d\tau\, w(\tau) \int_{-\infty}^{\infty} dp_\tau\, P(p_\tau) f(\bar{x}_\tau + p_\tau, \tau), \tag{3.7}$$

$$P(p_\tau) = \frac{1}{\sqrt{2\pi\nu_\tau^2}} \exp(-p_\tau^2/2\nu_\tau^2) \tag{3.8}$$

and

$$\nu_\tau^2 = \sigma^2\varepsilon(1-\tau)\tau, \quad \bar{x}_\tau = \tau(x_n - x_{n-1}) + x_{n-1}, \quad \tau = (s - s_{n-1})/\varepsilon. \tag{3.9}$$

Equation (3.7) is a key equation as it contains the partial averaging. The origin of α is the first cumulant in the expansion (3.5) and its evaluation will lead to a short-time propagator correct to second order in ε. Expanding α to order ε is consistent with the truncation of the higher cumulants in (3.5). The origin of β is the second cumulant which is formally calculated in appendix C. All we need to know here is that β is of order $\sigma^2\varepsilon$, since it will be set to zero at the end. We keep it to determine the order of the first correction term due to the truncation of all higher cumulants. After combining (3.6),(3.3) and (2.9) and performing the integration over k, we find that the joint PDF becomes

$$\begin{aligned} P(x_u, \mathcal{I}|x_t) \;\simeq\; & \int_{-\infty}^{\infty} dx_{N-1}...dx_1\, P[x_N, .., x_1|x_0] \\ & \times\; \exp\left[-\frac{\left(\mathcal{I} - \varepsilon\sum_{n=1}^{N}\alpha(x_n, x_{n-1}; \varepsilon)\right)^2}{2\varepsilon^2\sum_{n=1}^{N}\beta(x_n, x_{n-1}; \varepsilon)}\right] \left(2\pi\varepsilon^2\sum_{n=1}^{N}\beta(x_n, x_{n-1}; \varepsilon)\right)^{-1/2} \end{aligned} \tag{3.10}$$

where $x_N \equiv x_u$, $x_t \equiv x_0$ and the discrete path PDF is given by

$$P[x_N, ..., x_1|x_0] = \left(\frac{1}{2\pi\sigma^2\varepsilon}\right)^{\frac{N}{2}} \exp\left[-\sum_{n=1}^{N}\frac{(x_n - x_{n-1} - \mu\varepsilon)^2}{2\sigma^2\varepsilon}\right]. \tag{3.11}$$

Equation (3.11) describes the probability density of realizing a particular discrete path of the risk-neutral stochastic process (2.2). In the limit that β tends to zero, the joint PDF (3.10) becomes a delta function in $\mathcal{I}$ and we can show, using (2.8), that the option price becomes

$$C_F(S_t,t) \simeq e^{-rT}\int_{-\infty}^{\infty} dx_N ... dx_1 P[x_N,...,x_1|x_0]\; F\Big(e^{x_N}, \varepsilon\sum_{n=1}^{N}\alpha(x_n,x_{n-1};\varepsilon)\Big) + o(\varepsilon^2\sigma^2 T). \tag{3.12}$$

The order of the correction term follows from (3.10) and a saddle point expansion of the Gaussian integral over $\mathcal{I}$ in (2.8).

Equation (3.12) is the major result of this paper. The multi-dimensional integral can be evaluated by deterministic methods or more generally by standard Monte Carlo methods. In this case we use the observation that (3.12) is equivalent to an expectation of the option payoff function F, with respect to the discretely sampled risk-neutral diffusion process defined by (3.11), or equivalently by (2.2). In (3.12), the discretization time-scale ε is independent of any discrete option sampling time-scale. If we choose ε to match the interval between discrete option sampling, we find that (3.7) will reduce to a primitive short-time propagator and (3.12) will become

$$C_F(S_t,t) = e^{-rT}\int_{-\infty}^{\infty} dx_N ... dx_1 P[x_N,...,x_1|x_0] F\left(e^{x_N}, \sum_{n=0}^{N} w_n f(x_n)\right). \tag{3.13}$$

This is equivalent to a direct discretization of (2.7), consistent with a discretely sampled path dependent random variable described by (2.5) and (2.6). In this case no analytical partial averaging has been performed and the Monte Carlo evaluation of (3.13) is completely standard. The great advantage of the partial averaging method is that in (3.12), the discrete time interval ε can be chosen to be much larger than the option sampling time-scale. The partial averaging is performed in (3.7) where we must average over Gaussian fluctuations about the straight line path $\bar{x}_\tau$ connecting x_n and x_{n-1}. This corresponds to averaging over the high frequency fluctuations of the risk-neutral diffusion process. In practice, as will be seen in the next section, the partial averaging integral is easily performed analytically. It is simply the Gaussian transform of the function f, which defines the path dependent random variable in question via (2.5). Of most practical interest will be discretely sampled path dependent options, where $w(s)$ is given by (2.6). In this case the subsequent integral over τ in (3.7) reduces to a discrete sum which presents no problems.

For the special case $N = 1$, for which $\varepsilon = T$, $x_1 \equiv x_u$ and $x_0 \equiv x_t$, we find that (3.12) becomes

$$C_F(S_t,t) \simeq \frac{e^{-rT}}{\sqrt{2\pi\sigma^2 T}}\int_{-\infty}^{\infty} dx_u \,\exp\left[-\frac{(x_u - x_t - \mu T)^2}{2\sigma^2 T}\right] F\Big(e^{x_u}, T\alpha(x_u,x_t;T)\Big) + o(\sigma^2 T^3). \tag{3.14}$$

This describes rather generally an approximate path dependent option price. It can be simply evaluated as a one dimensional numerical integral. As a measure of the accuracy of (3.14), we ask at what time to maturity T, does the error term in (3.14) become 1% of the true option price. We assume a typical market volatility of $\sigma = 0.25$ and that the unknown coefficient of the correction term is equal to the true option price. We find that $T \simeq 0.5$ gives a 1% error, while $T \simeq 0.25$ gives an error of approximately 0.1%. These estimates begin to illustrate the power of the method presented here.

4 Examples

The previous formulation is rather general and can be applied to a range of path dependent options. In this section we will show how two important and qualitatively different classes of path dependent options fit into this framework.

4.1 Average rate options

The payoff of the geometric Asian option is some function of the path dependent random variable given by

$$\mathcal{I} = \int_t^u ds\, w(s) x_s \;, \tag{4.1}$$

where x_s is related to the risk-neutral asset price by (2.3). From (2.5) we can identify f with x_s. After performing the partial averaging (3.7), we find

$$\alpha(x_n, x_{n-1}; \varepsilon) = \int_0^1 d\tau\, w(\tau) \bar{x}_\tau. \tag{4.2}$$

For the continuous sampling ($w(\tau) = 1$) we find

$$\alpha(x_n, x_{n-1}; \varepsilon) = (x_n + x_{n-1})/2. \tag{4.3}$$

For this simple example α is just the primitive short-time propagator.

The payoff of the arithmetic Asian option will be some function of the path dependent random variable

$$\mathcal{I} = \int_t^u ds\, w(s) e^{x_s}. \tag{4.4}$$

From (2.5) we identify f with e^{x_s}. After performing the partial averaging (3.7) we find

$$\alpha(x_n, x_{n-1}; \varepsilon) = \int_0^1 d\tau\, w(\tau) e^{\bar{x}_\tau + \nu_\tau^2/2}. \tag{4.5}$$

We can expand (4.5) to order $\sigma^2\varepsilon$ without a significant loss of accuracy. This is consistent with the order of the truncation of the cumulant expansion (3.5). We then find

$$\alpha(x_n, x_{n-1}; \varepsilon) \simeq \int_0^1 d\tau\, w(\tau) e^{\bar{x}_\tau} \left(1 + \sigma^2\varepsilon(1-\tau)\tau/2 + o(\sigma^4\varepsilon^2)\right). \tag{4.6}$$

The final result will depend on whether we use discrete or continuous sampling. For continuous sampling we have $w(\tau) = 1$ and (4.6) becomes

$$\alpha(x_n, x_{n-1}; \varepsilon) \simeq e^{x_{n-1}} \left[\frac{1}{a}\Big(e^a - 1\Big) + \frac{\sigma^2\varepsilon}{2a^3}\Big(e^a(a-2) + a + 2\Big) + o(\sigma^4\varepsilon^2)\right], \tag{4.7}$$

where $a = x_n - x_{n-1}$. For discrete sampling we can perform the necessary summations analytically so the method is equally effective. It is instructive to compare (4.7) with the α that generates the primitive short-time propagator. This is obtained by approximating (3.7) by

$$\alpha(x_n, x_{n-1}; \varepsilon) \simeq \frac{1}{2}\Big(f(x_n) + f(x_{n-1})\Big). \tag{4.8}$$

One can see that (4.7) will only reduce to (4.8) in the limit $a \to 0$ (when $f = e^{x_s}$).

We can now use (3.12) to perform a Monte Carlo evaluation of the continuously sampled Asian option. Using (4.7) allows us to obtain accurate results by simulating only relatively low dimensional random paths of (2.2). This will avoid the problems associated with the Monte Carlo evaluation of these options [33].

4.2 Occupation time derivatives

A large and important class of path dependent options are those where the payoff depends on the time the asset price spends within a given region. This class of options have been referred to as occupation time derivatives and they have been discussed in detail by Linetsky [34, 35] and Hugonnier [36]. The valuation of debt and contingent claims with default risk can also be placed in this class [37, 38].

As a simple example, consider an option with a payoff that depends on the time spent below a possibly time dependent barrier B_s. This time is a path dependent random variable which can be expressed as (and similarly for the above case)

$$\mathcal{I} = \int_t^u ds\, w(s) H(B_s - e^{x_s}), \tag{4.9}$$

where H is a simple step function. We can therefore identify H with the function f in (2.5). After performing the partial averaging (3.7) we find

$$\alpha(x_n, x_{n-1}; \varepsilon) = \int_0^1 d\tau\, w(\tau) N\Big((\ln B_s - \bar{x}_\tau)/\nu_\tau\Big), \tag{4.10}$$

where N is the cumulative normal distribution function. This function is difficult to integrate analytically. However the real case of practical interest is discrete sampling. In this case the integration reduces to a discrete summation and presents no problems. Barrier options are the most basic and well known example of options that fall under this framework. An interesting generalization is the step option which has a finite knock-out rate. This is motivated by risk-management arguments and a variety of possible payoff functions have been discussed [34, 35]. All variations are included in this framework since the payoff function is an arbitrary function of the path dependent random variable.

Consider the example of derivatives depending on the occupation time between two barriers B_1 and B_2.[1] Included in this class are double barrier options [39] and the range products such as range notes and corridor options [40, 41]. This time is a path dependent random variable which can be expressed as

$$\mathcal{I} = \int_t^u ds\, w(s) \Big[H(B_2 - e^{x_s}) - H(B_1 - e^{x_s})\Big]. \tag{4.11}$$

From (2.5) we can identify the function f with the difference of two step functions. After performing the partial averaging (3.7) we find

$$\alpha(x_n, x_{n-1}; \varepsilon) = \int_0^1 d\tau\, w(\tau) \Big[N\Big((\ln B_2 - \bar{x}_\tau)/\nu_\tau\Big) - N\Big((\ln B_1 - \bar{x}_\tau)/\nu_\tau\Big)\Big]. \tag{4.12}$$

[1] the occupation time outside these barriers can be trivially constructed from this time

Our framework is easily generalized to include derivatives dependent on several occupation times.

5 Discussion and Conclusion

The aim of this paper was to present a new approach to evaluating the price of path dependent options. We considered options with the general payoff function (2.4), contingent on a path dependent random variable expressible in the form (2.5). The key results of this paper were the general evaluation formula (3.12) and the short time to expiry approximation (3.14). They give the option price after performing analytically a partial averaging, defined by equations (3.7-9), over the underlying risk-neutral diffusion process (2.2). Since the method is analytically based, it also gives the order of the error made by choosing a finite time discretization. Specific examples were presented in section 4.

In (3.12), the partial averaging allows one to choose the discrete time interval ε to be much larger than the option sampling time-scale. We could, for example, imagine choosing ε to be one month for an option with daily sampling. Clearly, the partial averaging method can greatly reduce the dimension of the integral in (3.12). This integral can be evaluated most generally by standard Monte Carlo methods. In this case the partial averaging will greatly reduce the dimension of the random paths to be simulated. In effect, it allows random simulations to be replaced with deterministic calculations. Standard methods to increase the efficiency of the Monte Carlo evaluation can still be used. These include variance reduction techniques [4], the simulation of sample paths using the Brownian bridge process [42] or the use of quasi Monte Carlo sampling [4, 43]. Interestingly, quasi Monte Carlo sampling is known to be more advantageous for low dimensional numerical integrals. The partial averaging method can therefore increase the relative gains made by the implementation of quasi Monte Carlo methods.

The framework presented here can be easily extended to multi-factor path dependent options [18, 22] and to path dependent options dependent on several path dependent random variables. Path integral methods have long been developed and used as a computational tool in theoretical and chemical physics. Hopefully the work presented here will stimulate more interest in the application of these methods to problems in computational finance.

A Path Integral Representation

Consider a stochastic process x_s, which obeys the stochastic differential equation [2]

$$dx_t = -g'(x_t)dt + \sigma dW_t. \tag{A.1}$$

In our notation $x_s \equiv x(s)$. We will rewrite this equation as

$$\dot{x}_t = -g'(x_t) + \sigma\zeta_t \tag{A.2}$$

[2] Note that any one-dimensional risk-neutral diffusion process can be cast into this form by a change of variable

where $\zeta_t = \frac{dW_t}{dt}$ and a dot denotes the derivative with respect to time. This notation is more suited for the path integral formulation. In (A.1) we assume that σ in independent of x_t (additive noise). The path integral formulation of the more general case (multiplicative noise) can be found in [44].

Consider a general continuous Gaussian noise process $\chi(s)$ (note that small s denotes the time history variable between time t and u and should not be confused with the price S_t). We discretize this process by defining a discrete time $s_n = n\varepsilon + t$, where $n = 0, 1, ..., N$ and $\varepsilon = T/N$ with $T = u - t$. Note that ε is equivalent to ds when $N \to \infty$. The now discrete Gaussian process is fully defined by the normalized probability density functional

$$\mathcal{P}\left[\chi_1, ..., \chi_N\right] = (2\pi)^{-n/2} \left(\det R\right)^{-1/2} \exp\left[-\frac{1}{2}\sum_{n,m=1}^{N} \chi_n\, R^{-1}_{nm}\, \chi_m\right], \tag{A.3}$$

where $\chi_n = \chi(s_n)$, $E[\chi_n \chi_m] = R_{nm}$ and R^{-1}_{nm} denotes the inverse matrix. Equation (A.3) fully defines the probability density of the whole history of the discrete Gaussian process. This normalization condition implies

$$\int_{-\infty}^{\infty} \mathcal{P}\left[\chi_1, ..., \chi_N\right]\, d\chi_1 ... d\chi_N = 1. \tag{A.4}$$

Since dW_t has a variance dt, we know ζ_t will have a variance dt^{-1}. We then find that for the discrete time white noise process ζ_n, we have $E[\zeta_n \zeta_m] = \varepsilon^{-1}\delta_{nm}$ where δ_{nm} is the unit diagonal matrix. Using (A.3) we then find

$$\mathcal{P}\left[\zeta_1, ..., \zeta_N\right] = \left(\frac{\varepsilon}{2\pi}\right)^{N/2} \exp\left[-\frac{1}{2}\sum_{n=1}^{N} \varepsilon\, \zeta_n^2\right]. \tag{A.5}$$

In the continuous limit this becomes

$$\mathcal{P}[\zeta_s] = \left(\frac{\varepsilon}{2\pi}\right)^{N/2} \exp\left[-\frac{1}{2}\int_t^u ds\ \zeta_s^2\right], \tag{A.6}$$

where the continuous time expressions are written with the understanding that $\varepsilon \to 0$ and $N \to \infty$ such that $N\varepsilon = T$.

We wish to use the probability density functional (A.6) to find a probability density functional for the stochastic process x_s. To do this we need to first discretize in time (A.2). In discrete time (A.2) becomes

$$\frac{x_n - x_{n-1}}{\varepsilon} = -g'(\tilde{x}_n) + \sigma\zeta_n, \tag{A.7}$$

where

$$\tilde{x}_n = \phi x_n + (1-\phi) x_{n-1} \tag{A.8}$$

and ϕ is a discretization parameter between 0 and 1. From (A.7) we see that a path $\{\zeta_1, ..., \zeta_N\}$ maps to a unique path $\{x_1, ..., x_N\}$ *as long as* x_0 *is given*. We will therefore write, by virtue of (A.4)

$$\int_{-\infty}^{\infty} \mathcal{P}[x_1, ..., x_N | x_0]\, \mathcal{J}\, dx_1 ... dx_N = 1, \tag{A.9}$$

where $\mathcal{J}$, defined by

$$\mathcal{J} = \det \left| \frac{\partial \zeta_n}{\partial x_m} \right|, \quad m, n = 1, ..., N \tag{A.10}$$

is the Jacobian of the change in coordinates. In the continuous limit we can substitute (A.2) into (A.6) to obtain

$$\mathcal{P}\left[x_s\right] = \left(\frac{\varepsilon}{2\pi}\right)^{N/2} \exp\left[-\frac{1}{2\sigma^2}\int_t^u ds\ \left(\dot{x}_s + g'(x_s)\right)^2\right]. \tag{A.11}$$

We show in appendix B that the Jacobian (A.10) becomes in the continuous limit

$$\mathcal{J} = (\sigma\varepsilon)^{-N} \exp\left[\phi \int_t^u ds\ g''\left(x_s\right)\right]. \tag{A.12}$$

The only sensible choice is $\phi = 1/2$ [44]. Combining the Jacobian and (A.11) we obtain a new probability density functional

$$\mathcal{P}_x\left[x_s\right] = \exp\left[-\frac{1}{2\sigma^2}\int_t^u ds\ \left(\dot{x}_s + g'(x_s)\right)^2 + \frac{1}{2}\int_t^u ds\ g''(x_s)\right], \tag{A.13}$$

which is normalized with respect to the functional measure $\mathcal{D}_s$ which is the continuous limit of

$$\mathcal{D}x_s = \left(2\pi\varepsilon\sigma^2\right)^{-N/2} dx_1 ... dx_{N-1}. \tag{A.14}$$

This means the conditional probability density function (PDF) of (A.2) is given by the path integral

$$P(x_u, x_t) = \int_{x_t}^{x_u} \mathcal{D}x_s\, \mathcal{P}_x[x_s]. \tag{A.15}$$

The expectation of a functional $\mathcal{F}[x_s]$, conditional on the initial value x_t, can now be expressed in the path integral form

$$E_{x_t}\Big[\mathcal{F}[x_s]\Big] = \int_{-\infty}^{\infty} dx_u \int_{x_t}^{x_u} \mathcal{D}x_s\, \mathcal{P}_x[x_s]\, \mathcal{F}[x_s]. \tag{A.16}$$

Using (2.7), we can now write the option price in the path integral form

$$C_F(S_t, t) = e^{-rT} \int_{-\infty}^{\infty} dx_u \int_{x_t}^{x_u} \mathcal{D}x_s\, \mathcal{P}_x[x_s]\, F(e^{x_u}, \mathcal{I}). \tag{A.17}$$

We will not deal directly with the above path integral representation of the option price. From (2.8) we see that we can extract the payoff function out of the path integral if we instead focus on the path integral representation of the joint PDF $P(x_u, \mathcal{I}|x_t)$. In the next subsection we will see how to calculate this function.

A.1 Calculating the joint PDF

The joint PDF introduced in (2.8) can be simply expressed as the path integral

$$P(x_u, \mathcal{I}|x_t) = \int_{x_t}^{x_u} \mathcal{D}\hat{x}_s \, \mathcal{P}_x[\hat{x}_s] \delta(\mathcal{I} - \hat{\mathcal{I}}). \tag{A.18}$$

Using the Fourier representation of the delta function and (2.5), we find that (A.18) becomes

$$P(x_u, \mathcal{I}|x_t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ik\mathcal{I}} \int_{x_t}^{x_u} \mathcal{D}\hat{x}_s \, \mathcal{P}_x[\hat{x}_s] \exp\left[ik \int_t^u ds \, w(s) f(\hat{x}_s, s)\right]. \tag{A.19}$$

Substituting (A.13) into (A.19) and using

$$\int_t^u ds \, \dot{x}_s \, g'(x_s) = g(x_u) - g(x_t), \tag{A.20}$$

we find that (A.19) becomes

$$P(x_u, \mathcal{I}|x_t) = \frac{1}{2\pi} \exp\left[\frac{g(x_t) - g(x_u)}{\sigma^2}\right] \int_{-\infty}^{\infty} dk e^{-ik\mathcal{I}} K(x_u, x_t; T), \tag{A.21}$$

where K, which we will refer to as the propagator, is defined by

$$K(x_u, x_t; T) = \int_{x_t}^{x_u} \mathcal{D}x_s \exp\left[-\frac{1}{2\sigma^2} \int_t^u ds \ \left[\dot{x}_s^2 + V(x_s, s)\right]\right] \tag{A.22}$$

and what we will call the potential function V is

$$V(x_s, s) = g'^2(x_s) - \sigma^2 \ g''(x_s) - 2ik\sigma^2 \, w(s) \, f(x_s, s). \tag{A.23}$$

Of special interest is the geometric Brownian motion model defined by (2.2). Comparing (2.2) with (A.1), we can identify $g'(x_t)$ with μ and we find the joint PDF (A.21) becomes (2.9) with the potential (2.11). Because the drift term is constant in this case, it can be extracted out of the path integral and we are left with an imaginary potential.

B Calculating the Jacobian

In this appendix we will calculate the Jacobian (A.10). In discrete time the stochastic differential equation (SDE) (A.2) becomes

$$\frac{x_{n-} x_{n-1}}{\varepsilon} = -g'(\tilde{x}_n) + \sigma(\tilde{x}_n)\zeta_n, \tag{B.1}$$

where it has been generalized to include a non-constant diffusion coefficient. From this discrete SDE we see that

$$\frac{\partial \zeta_n}{\partial x_m} = 0, \ \text{ for } \ m > n. \tag{B.2}$$

Therefore the Jacobian matrix is triangular and from (A.10) we obtain

$$\mathcal{J} = \prod_{n=1}^{N} \frac{\partial \zeta_n}{\partial x_n}. \tag{B.3}$$

From (B.1) we find

$$\frac{1}{\varepsilon}\frac{\partial x_n}{\partial \zeta_n} = -\phi g''(\tilde{x}_n)\frac{\partial x_n}{\partial \zeta_n} + \phi\, \zeta_n\, \sigma'(\tilde{x}_n)\frac{\partial x_n}{\partial \zeta_n} + \sigma(\tilde{x}_n), \tag{B.4}$$

where we have used

$$\frac{\partial \tilde{x}_n}{\partial \zeta_n} = \frac{\partial \tilde{x}_n}{\partial x_n}\frac{\partial x_n}{\partial \zeta_n} = \phi \frac{\partial x_n}{\partial \zeta_n} \tag{B.5}$$

obtained from (A.8). Multiplying (B.4) by $\varepsilon \frac{\partial \zeta_n}{\partial x_n}$ we obtain

$$\frac{\partial \zeta_n}{\partial x_n} = \frac{1 + \varepsilon\, \phi g''(\tilde{x}_n) - \varepsilon\, \phi\, \zeta_n \sigma'(\tilde{x}_n)}{\varepsilon \sigma(\tilde{x}_n)}. \tag{B.6}$$

For small ε (B.6) becomes

$$\begin{aligned} \frac{\partial x_n}{\partial \zeta_n} &\simeq \varepsilon \sigma(\tilde{x}_n)\Big(1 - \varepsilon\, \phi\, g''(\tilde{x}_n) + \varepsilon\, \phi\, \zeta_n \sigma'(\tilde{x}_n)\Big) \\ &\simeq \varepsilon \sigma(\tilde{x}_n) \exp\Big[\varepsilon\, \phi \Big(\sigma'(\tilde{x}_n)\zeta_n - g''(\tilde{x}_n)\Big)\Big]. \end{aligned} \tag{B.7}$$

Using this result we find

$$\prod_{n=1}^{N} \frac{\partial x_n}{\partial \zeta_n} \simeq \varepsilon^N \left(\prod_{n=1}^{N} \sigma(\tilde{x}_n)\right) \exp\left[\varepsilon\, \phi \sum_{n=1}^{N} \Big(\sigma'(\tilde{x}_n)\zeta_n - g''(\tilde{x}_n)\Big)\right]. \tag{B.8}$$

In the continuous limit $\varepsilon \to 0$, we therefore find

$$\mathcal{J}^{-1} = \varepsilon^N \left(\prod_{n=1}^{N} \sigma(\tilde{x}_n)\right) \exp\left[\phi \int_t^u ds \left(\sigma'(x_s)\zeta_s - g''(x_s)\right)\right]. \tag{B.9}$$

From the continuous SDE (A.2) we have

$$\zeta_s = \Big(\dot{x}_s + g'(x_s)\Big)/\sigma(x_s). \tag{B.10}$$

Substituting this into (B.9) we find

$$\mathcal{J} = \varepsilon^{-N} \left(\prod_{n=1}^{N} \sigma(\tilde{x}_n)\right)^{-1} \exp\left[-\phi \int_t^u ds \left((\dot{x}_s + g'(x_s))\sigma'(x_s)/\sigma(x_s) - g''(x_s)\right)\right]. \tag{B.11}$$

This agrees with equation 2.4.37 in Stratonovich [10] who derives the Jacobian for a general multi-factor system of SDE's. For a constant diffusion coefficient (B.11) reduces to (A.12).

C Improved Short-time Propagator

The goal of this appendix is to obtain an expansion of $K(x_u, x_t; T)$ in powers of T. The derivation here is inspired by the cumulant method used in connection with the quantum statistical density matrix [31].

We must first write the propagator in the Fourier path integral representation (FPIR). We decompose the paths as

$$x_\tau = \bar{x}_\tau + \left(\frac{4\sigma^2 T}{\pi}\right)^{1/2} \sum_{n=1}^{\infty} \frac{z_n \sin(n\pi\tau)}{n} \tag{C.1}$$

where

$$\bar{x}_\tau = \tau(x_u - x_t) + x_t, \quad \tau = (s-t)/T. \tag{C.2}$$

The term $\bar{x}_\tau$ in (C.1) is the straight line path connecting x_t and x_u. The remaining terms are the harmonic perturbations about the straight line path. With this we can now write

$$\frac{1}{2\sigma^2}\int_t^u ds\, \left[\dot{x}_s^2 + V(x_s, s)\right] = \frac{(x_u - x_t)^2}{2\sigma^2 T} + \pi \sum_{n=1}^{\infty} z_n^2 + \frac{T}{2\sigma^2}\int_0^1 d\tau\, V(x_\tau, \tau). \tag{C.3}$$

Summing over all paths between x_t and x_u is equivalent to integrating over all possible values of the Fourier coefficients $\{z_n\}$. This means we can write

$$\int_{x_t}^{x_u} \mathcal{D}x_s = (\text{constant}) \times \int_{-\infty}^{\infty} dz_1 ... dz_\infty, \tag{C.4}$$

where the constant is some Jacobian factor resulting from the change of integration variables. Substituting (C.4) and (C.3) into (2.10), we obtain the FPIR of the propagator

$$K(x_u, x_t; T) = K_f(x_u, x_t; T)\int_{-\infty}^{\infty} dz_1 ... dz_\infty \exp\left[-\pi\sum_{n=1}^{\infty} z_n^2 - \frac{T}{2\sigma^2}\int_0^1 d\tau\, V(x_\tau, \tau)\right] \tag{C.5}$$

where $K_f(x_u, x_t; T)$ is given by

$$K_f(x_u, x_t; T) = \left(\frac{1}{2\pi\sigma^2 T}\right)^{1/2} \exp\left(-\frac{(x_u - x_t)^2}{2\sigma^2 T}\right). \tag{C.6}$$

The constant Jacobian factor in (C.4) must equal the prefactor of (C.6) to obtain the correct propagator when $V = 0$ (clearly the Jacobian is independent of the potential). We can rewrite (C.5) as

$$K(x_u, x_t; T) = K_f(x_u, x_t; T)\, E\left[\exp\left(-\frac{T}{2\sigma^2}\int_0^1 d\tau\, V(x_\tau, \tau)\right)\right]_{\{z_n\}}, \tag{C.7}$$

where the expectation is with respect to the set of independent Gaussian random variables $\{z_n\}$ with zero mean and variance $1/2\pi$. The FPIR can also be set up using a reference potential that is quadratic [32].

We can use the FPIR (C.7) to derive an expansion in time for the short time propagator. We have using the cumulant expansion

$$K(x_u, x_t; T) = K_f(x_u, x_t; T) \exp\left(\sum_{m=1}^{\infty} \frac{1}{m!} \left(-\frac{T}{2\sigma^2} \right)^m C_m(x_u, x_t; T) \right), \tag{C.8}$$

where

$$C_1 = M_1,\ C_2 = M_2 - M_1^2,\ C_3 = M_3 - 3M_1 M_2 + 2M_1^3, ... \tag{C.9}$$

and

$$M_m = E\left[\left(\int_0^1 d\tau\, V(x_\tau, \tau) \right)^m \right]_{\{z_n\}}. \tag{C.10}$$

The propagator (C.8) has the general structure defined by (3.2) and (3.5). Consider the first cumulant which is

$$C_1(x_u, x_t; T) = \int_0^1 d\tau\, E\left[V(x_\tau, \tau) \right]_{\{z_n\}}. \tag{C.11}$$

We know from (C.1) that x_τ is a sum of Gaussian random variables which itself must be Gaussian distributed. So when calculating the expectation (C.11), we can replace x_τ by

$$x_\tau = \bar{x}_\tau + p_\tau, \tag{C.12}$$

where p_τ is a single Gaussian random variable with variance

$$\nu_\tau^2 = \frac{2\sigma^2 T}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2(n\pi\tau)}{n^2} = \sigma^2 T(1-\tau)\tau. \tag{C.13}$$

Using this result we find that

$$E\left[V(x_\tau, \tau) \right]_{\{z_n\}} = \int_{-\infty}^{\infty} dp_\tau\, P(p_\tau) V(\bar{x}_\tau + p_\tau, \tau) \tag{C.14}$$

where

$$P(p_\tau) = \frac{1}{\sqrt{2\pi\nu_\tau^2}} \exp(-p_\tau^2/2\nu_\tau^2). \tag{C.15}$$

The first cumulant then becomes

$$C_1(x_u, x_t; T) = \int_0^1 d\tau \int_{-\infty}^{\infty} dp_\tau\, P(p_\tau) V(\bar{x}_\tau + p_\tau, \tau). \tag{C.16}$$

Expanding the potential around $p_\tau = 0$ and performing the Gaussian integrals we obtain

$$E\left[V(x_\tau, \tau) \right]_{\{z_n\}} \simeq V(\bar{x}_\tau, \tau) + \frac{\nu_\tau^2}{2} V''(\bar{x}_\tau, \tau) + \frac{\nu_\tau^4}{8} V''''(\bar{x}_\tau, \tau) + o(T^3), \tag{C.17}$$

which from (C.13) is an expansion in time T.

Consider now the second cumulant

$$C_2(x_u, x_t; T) = \int_0^1 d\tau d\tau' \bigg(E\left[V(x_\tau,\tau)V(x_{\tau'},\tau')\right]_{\{z_n\}} - E\left[V(x_\tau,\tau)\right]_{\{z_n\}} E\left[V(x_{\tau'},\tau')\right]_{\{z_n\}} \bigg). \tag{C.18}$$

In (C.18) we can replace x_τ and $x_{\tau'}$ with

$$x_\tau = \bar{x}_\tau + p_\tau, \quad x_{\tau'} = \bar{x}_{\tau'} + p_{\tau'} \tag{C.19}$$

where p_τ and $p_{\tau'}$ are two correlated Gaussian random variables with variances

$$\nu_\tau^2 = \sigma^2 T(1-\tau)\tau, \quad \nu_{\tau'}^2 = \sigma^2 T(1-\tau')\tau'. \tag{C.20}$$

Using (C.1) we find the covariance is

$$c(\tau,\tau') = E[p_\tau p_{\tau'}] = \frac{4\sigma^2 T}{\pi} \sum_{n,n'=1}^{\infty} E[z_n z_{n'}]_{\{z_n\}} \frac{\sin(n\pi\tau)\sin(n'\pi\tau')}{nn'}. \tag{C.21}$$

Using

$$E[z_n z_{n'}]_{\{z_n\}} = \frac{1}{2\pi}\delta_{nn'} \tag{C.22}$$

and the identity

$$\frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\sin(n\pi\tau)\sin(n\pi\tau')}{n^2} = \tau_l(1-\tau_g), \tag{C.23}$$

we find

$$c(\tau,\tau') = \sigma^2 T\, \tau_l(1-\tau_g) \tag{C.24}$$

where τ_l is the lesser of τ and τ' and τ_g is the greater of τ and τ'. Using (A.3), we can write the probability distribution of p_τ and $p_{\tau'}$ as

$$P(p_\tau, p_{\tau'}) = \frac{1}{2\pi}\frac{1}{\sqrt{\nu_\tau^2\nu_{\tau'}^2 - c^2}} \exp\left[-\frac{(\nu_{\tau'}^2 p_\tau^2 - 2c\, p_\tau p_{\tau'} + \nu_\tau^2 p_{\tau'}^2)}{2(\nu_\tau^2\nu_{\tau'}^2 - c^2)}\right]. \tag{C.25}$$

We therefore find that

$$E\left[V(x_\tau,\tau)V(x_{\tau'},\tau')\right]_{\{z_n\}} = \int_{-\infty}^{\infty} dp_\tau dp_{\tau'} P(p_\tau,p_{\tau'}) V(\bar{x}_\tau + p_\tau, \tau) V(\bar{x}_{\tau'} + p_{\tau'}, \tau'). \tag{C.26}$$

Substituting this result and (C.14) into (C.18), we find that the second cumulant is

$$\begin{aligned} C_2(x_u, x_t; T) &= \int_0^1 d\tau d\tau' \left(\int_{-\infty}^{\infty} dp_\tau dp_{\tau'}\, P(p_\tau,p_{\tau'}) V(\bar{x}_\tau + p_\tau, \tau) V(\bar{x}_{\tau'} + p_{\tau'}, \tau') \right. \\ &- \left. \int_{-\infty}^{\infty} dp_\tau dp_{\tau'}\, P(p_\tau) P(p_{\tau'}) V(\bar{x}_\tau + p_\tau, \tau) V(\bar{x}_{\tau'} + p_{\tau'}, \tau') \right). \end{aligned} \tag{C.27}$$

Expanding the second cumulant around $p = 0$ we find that the first order remaining term is

$$C_2(x_u, x_t; T) \simeq \sigma^2 T \int_0^1 d\tau d\tau'\, V'(\bar{x}_\tau, \tau) V'(\bar{x}_{\tau'}, \tau')\, \tau_l(1-\tau_g) + o(\sigma^4 T^2). \tag{C.28}$$

This follows since the integral we have to do is just the covariance c.

It can be shown that the mth cumulant is of order $(\sigma^2 T)^{m-1}$. Therefore from (C.8) we see that C_2 starts contributing at order T^3, while C_3 starts contributing at order T^5. This means that to get the expansion (C.8) correct to order T^3, we need only expand C_1 to order T^2 and C_2 to order T as has been done in (C.17) and (C.28). This means that C_1 alone will give the correct propagator to order T^2. Substituting (C.28) and (C.17) into (C.8) we find

$$\begin{aligned} K(x_u, x_t; T) \simeq\ & K_f(x_u, x_t; T) \exp\left[-\frac{T}{2\sigma^2}\int_0^1 d\tau\, V(\bar{x}_\tau, \tau) - \frac{T^2}{4}\int_0^1 d\tau\, \tau(1-\tau)\, V''(\bar{x}_\tau, \tau)\right. \\ & - \frac{\sigma^2 T^3}{16}\int_0^1 d\tau\, \tau^2(1-\tau)^2 V''''(\bar{x}_\tau, \tau) \\ & \left. + \frac{T^3}{8\sigma^2}\int_0^1 d\tau d\tau'\, V'(\bar{x}_\tau, \tau) V'(\bar{x}_{\tau'}, \tau')\, \tau_l(1-\tau_g) + o(T^4)\right], \end{aligned} \tag{C.29}$$

which is our desired expansion of the propagator to third order in time.

Acknowledgements: I would like to thank the Australia Research Council for their generous support of this research through an Australian Postdoctoral Research Fellowship. I would also like to thank Science & Finance and the School of Economics and Finance, University of Technology, Sydney, where part of this research was carried out.

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