Hedging large risks reduces the transaction costs

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Abstract

As soon as one accepts to abandon the zero-risk paradigm of Black-Scholes, very interesting issues concerning risk control arise because different definitions of the risk become unequivalent. Optimal hedges then depend on the quantity one wishes to minimize. We show that a definition of the risk more sensitive to the extreme events generically leads to a decrease both of the probability of extreme losses and of the sensitivity of the hedge on the price of the underlying (the 'Gamma'). Therefore, the transaction costs and the impact of hedging on the price dynamics of the underlying are reduced.

It is well known that the perfect Black-Scholes hedge only works in the ideal case of a continuous time, log-Brownian evolution of the price of the underlying. Unfortunately, this model is rather remote from reality: the distribution of price changes has 'fat tails', which persist even for rather long time lags (see, e.g. [1, 2, 3]). This makes the whole idea of zero-risk strategies and perfect replication shady. An alternative view was proposed in [4, 3, 5], where one accepts from the start that the risk associated with option trading is in general non zero, but can be *minimized* by adopting an appropriate hedging strategy. If the risk is defined as the variance of the global wealth balance, as was proposed in [4, 3, 5], one can obtain a expression for the optimal hedge that is valid under rather mild assumptions on the dynamics of price changes. This optimal strategy allows one to compute the 'residual' risk, which is in general non zero, and actually rather large in practice. For typical one month at the money options, the minimal standard deviation of the wealth balance is of the order of a third of the option price itself! This more general theory allows one to recover all the Black-Scholes results in the special case of Gaussian returns in the continuous time limit, in particular the well known 'Δ-hedge', which states that the optimal strategy is the derivative of the option price with respect to the underlying.

However, as soon as the risk is non zero, the various possible definitions of 'risk' become unequivalent. One can for example define the risk through a higher moment of the wealth balance distribution – for example the fourth moment (whereas the variance is the second moment). This is interesting since higher moments are more sensitive to extreme events. The minimisation of the fourth moment of the distribution therefore allows one to reduce the probability of large losses, which is indeed often a concern to risk managers. One could also consider the 'Value-at-Risk' (defined as the loss level with a certain small probability) as the relevant measure of large risks, and aim at minimizing that quantity: this is a generalisation of the present work which is still in progress [6]. However, our main conclusions remain valid for this case as well.

Our results can be summarized as follows: the optimal strategy obtained using the fourth moment as a measure of risk varies much less with the moneyness of the underlying than both the Black-Scholes Δ-hedge and the optimal variance hedge. This is very interesting because it means that when the price of the underlying changes, the corresponding change in the hedge position is reduced. Therefore, the transaction costs associated to option hedging decrease as one attemps to hedge away large risks. Our numerical estimates show that this reduction is substantial. This result is also important for the global stability of markets: it is well known that the hedging strategies can feedback on the dynamics of the markets, as happened during the crash of October 1987, where the massive use of the Black-Scholes hedge (through 'Insurance Portfolio' strategies) amplified the drop of the market. Therefore, part of the 'fat-tails' observed in the dynamics of price changes can be attributed to this non-linear feedback effect. By reducing the sensitivity of the hedge on the price of the underlying, one can also hope to reduce this destabilising feedback.

Let us present our mathematical and numerical results in a rather cursory way (a more detailed version will be published separately [7]). In order to keep the discussion simple, we will assume that the interest rate is zero. In this case, the global wealth balance ΔW associated to the writing of a plain vanilla European call option can be written as:

$$\Delta W = \mathcal{C} - \max(x_N - x_s, 0) + \sum_{i=1}^{N-1} \phi_i(x_i)[x_{i+1} - x_i], \qquad N = \frac{T}{\tau} \qquad (1)$$

where $\mathcal{C}$ is the option premium, x_i the price of the underlying at time $t = i\tau$, $\phi(x)$ the hedging strategy, T the maturity of the option, x_s the strike and τ the time interval between rehedging. Previous studies focused on the risk defined as $\mathcal{R}_2 = \langle \Delta W^2 \rangle$, while the fair game option premium is such that $\langle \Delta W \rangle = 0$ (here, $\langle ... \rangle$ means an average over the *historical* distribution). As stated above, this allows one to recover precisely the standard Black-Scholes results if the statistics of price returns is Gaussian and one lets τ tend to 0 (continuous time limit). This is shown in full detail in [3].

Here, we consider as an alternative measure of the risk the quantity $\mathcal{R}_4 = \langle \Delta W^4 \rangle$. The corresponding optimal hedge is such that the functional derivative of $\mathcal{R}_4$ with respect to $\phi_i(x)$ is zero. This leads to a rather involved cubic equation on $\phi_i(x)$ (whereas the minimisation of $\mathcal{R}_2$ leads to a simple linear equation on $\phi_i(x)$). Further insight can be gained by first assuming a time *independent* strategy, i.e. $\phi_i(x) \equiv \phi_0(x)$. The corresponding cubic equation only depends on the terminal price distribution and can then be solved explicitely, leading to a unique real solution ϕ_4^* between 0 and 1. We show in Fig. 1 the evolution of the optimal strategy ϕ_4^* as a function of the moneyness, in the case where the terminal distribution is a symmetric exponential (which is often a good description of financial data), for $T = 1$ month and a daily volatility of 1%. The corresponding price of the at-the-money call is 2.73. On the same figure, we have also plotted the Black-Scholes Δ-hedge, and the hedge ϕ_2^* corresponding to the minimisation of $\mathcal{R}_2$. All these strategies vary between zero for deeply out of the money options to one for deeply in the money options, which is expected. However, as mentioned above, the variation of ϕ^* with moneyness is much weaker when $\mathcal{R}_4$ is chosen as the measure of risk. For example, for in the money options (resp. out of the

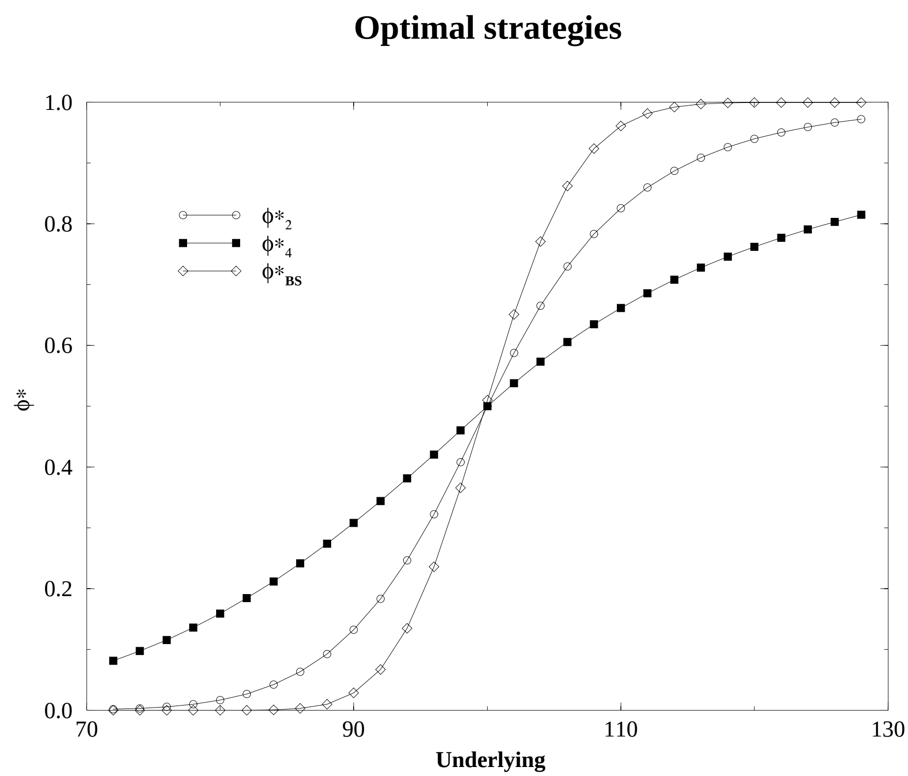

Figure 1: Three different strategies: ϕ_2^* minimizes the variance of the wealth balance, ϕ_4^* minimizes its fourth moment, whereas ϕ_{BS}^* is the Black-Scholes Δ-hedge. The strike price is 100, the maturity equal to one month, the daily volatility is 1% and the terminal price distribution is assumed to be a symmetric exponential, with an excess kurtosis of 3. The three strategies are equal to 1/2 at the money. Note that ϕ_4^* varies much less than the other two with moneyness.

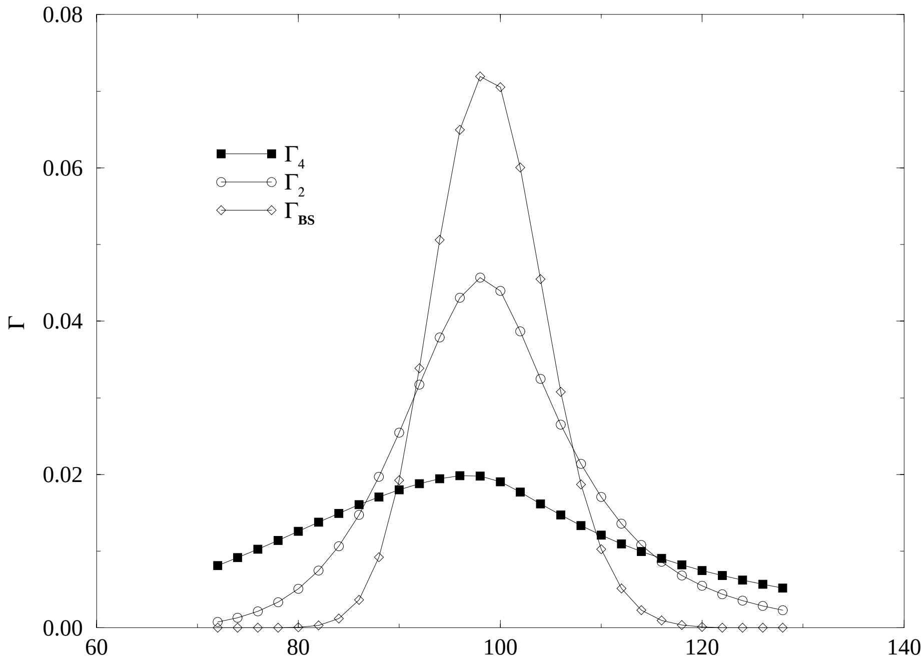

Figure 2: The three corresponding 'Gamma's', defined as the derivative of the strategy ϕ^* with respect to the price of the underlying. This quantity is important since the transaction costs for at the money options are proportional to $\Gamma(100)$.

money), ϕ_4^* is smaller (resp. greater) than the Black-Scholes Δ or than ϕ_2^*. This is because a possible large drop of the stock, which would suddenly drive the option out of the money and therefore lead to large losses due to the long position on stocks, is better taken into account by considering $\mathcal{R}_4$. One can illustrate this result differently by plotting the derivative of ϕ^* with respect to the price of the stock, which is the 'Gamma' of the option – see Fig. 2. One sees that in our example the at-the-money Gamma is decreased by a factor 3.5 compared to the Black-Scholes Gamma. The corresponding average transaction costs for rehedging are therefore also expected to decrease by the same amount.

It is interesting to study the full probability distribution function of ΔW for the following three cases: unhedged, hedged *à la* Black-Scholes or hedged following ϕ_4^*. Of particular interest is the probability p of large losses – for example, the probability of losing a certain amount $|\Delta W^*|$, defined as:

$$p = \int_{-\infty}^{-|\Delta W^*|} d\Delta W \; P(\Delta W) \tag{2}$$

The results for $|\Delta W^*| = 10$ (which is four times the option premium) are shown on Fig. 3 for different values of the strike price. One sees that even in the restrictive framework of a purely static hedge, ϕ_4^* allows one to decrease substantially the probability of large losses. For $x_s = 110$, this probability is decreased by a factor 3 to 4 as compared to the Black-Scholes hedge! For at-the-money options, since the static strategies are identical ($\phi_{BS}^* = \phi_4^* = 1/2$), one finds no difference in p.

We have up to now considered the simple case of a purely static strategy. In the case of the minimisation of $\mathcal{R}_2$, one can show that the fully dynamical hedge can be obtained by a simple time translation of the static one, that is, one can compute ϕ_{2i}^* by again assuming a static hedge, but with an initial time translated from 0 to $t = i\tau$. This can be traced back to the fact that if the price increments are uncorrelated (but not necessarily independent), the variance of the total wealth balance is the sum of the variances of the 'instantaneous' wealth balances $\Delta W_i = W_{i+1} - W_i$. This is no longer true if one wants to minimise $\mathcal{R}_4$. However, we have shown for $N = 2$ that the simple 'translated' strategy ϕ_4^* is numerically very close to (but different from) the true optimum. Since we are in the neighbourhood of a quadratic minimum, an error of order ϵ on the strategy will only increase the risk to order ϵ^2 and is therefore often completely negligible. [Note that a similar argument also holds in the case of ϕ_2^*: even if the Black-Scholes Δ is in general different from ϕ_2^*, the difference is often small and leads to a very small increase of the risk – see the discussion in [3]].

Finally, it is important to note that the optimal ϕ_4^* hedge for a book of options on the same underlying is not the simple linear superposition of the optimal hedge for the individual options in the book, whereas this is indeed the correct result for variance hedging. However, we have found in the case of a book containing two options with different strikes but the same maturity, that the difference between the optimal hedge and the simple linear prescription is again numerically very small.

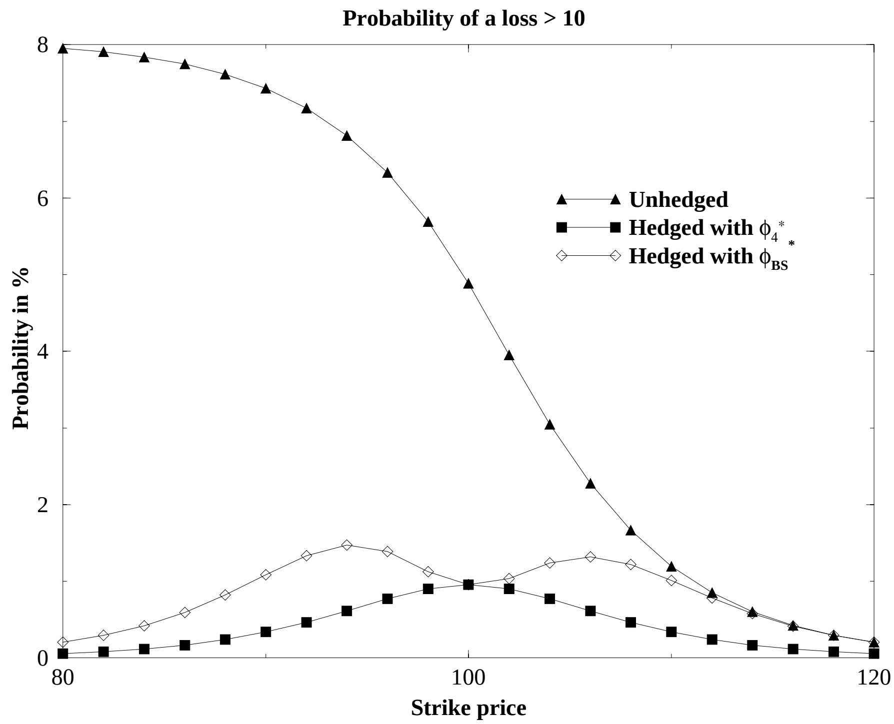

Figure 3: The probability p of losing four times the premium, as a function of the strike price, for the three different strategies: unhedged, Black-Scholes, and ϕ_4^*. Note the substantial decrease of p for out and in the money options, even in this restrictive case where the strategy is purely static.

As a conclusion, we hope to have convinced the reader that as soon as one accepts to abandon the zero-risk paradigm of Black-Scholes, very interesting issues concerning risk control arise because different definitions of the risk become unequivalent. [In the Black-Scholes world, the risk is zero, whatever the definition of risk !] Therefore, optimal hedges depend on the quantity one wishes to minimize. We have shown here that a definition of the risk more sensitive to the extreme events generically leads to a decrease of the sensitivity of the hedge on the price of the underlying (the 'Gamma'). Therefore, both the transaction costs and the impact of hedging on the price dynamics of the underlying are reduced.

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