



ELSEVIER

Physica A 313 (2002) 238–251

PHYSICA A

www.elsevier.com/locate/physa

# An introduction to statistical finance

Jean-Philippe Bouchaud<sup>a,b,\*</sup>

<sup>a</sup>*Service de Physique de l'État Condensé, Centre d'études de Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France*

<sup>b</sup>*Science & Finance, Capital Fund Management, 109-111 rue Victor-Hugo, 92532 France*

---

## Abstract

We summarize recent research in a rapid growing field, that of statistical finance, also called 'econophysics'. There are three main themes in this activity: (i) empirical studies and the discovery of interesting universal features in the statistical texture of financial time series, (ii) the use of these empirical results to devise better models of risk and derivative pricing, of direct interest for the financial industry, and (iii) the study of 'agent-based models' in order to unveil the basic mechanisms that are responsible for the statistical 'anomalies' observed in financial time series. We give a brief overview of some of the results in these three directions. © 2002 Elsevier Science B.V. All rights reserved.

---

## 1. Introduction

The last 10 years have witnessed very significant changes in finance, both as an academic subject and as a professional field. Finance is becoming an empirical (rather than axiomatic) science; correspondingly, financial engineers (quants) have an increasingly important role to play in the financial industry. There are various reasons for this change, but the most important must surely be the availability of data, and the possibility to access and process this data very quickly. Fifteen years ago, it was not so easy to find data, even on the most liquid markets. Now, one has direct access to high-frequency data (on the scale of seconds) not only for stocks, currencies or interest rates, but also for more exotic markets such as option markets, energy markets, weather derivatives, etc. This means that any statistical model, or theoretical idea, can and must be tested against available data, as physicists are (probably better than other

---

\* Corresponding author. Service de Physique de l'État Condensé, Centre d'études de Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France.

*E-mail address:* bouchau@drecam.saclay.cea.fr (J.-P. Bouchaud).

communities) trained to do. When data was scarce, logical consistency and simplicity were the only guides; not looking at data nowadays is just an excuse for sparing mathematically beautiful, but often inadapted models. Correspondingly, quants in banks are asked to ‘back-test’ trading strategies, option prices or risk estimates against past data, and understand in detail any discrepancies or artefacts.

From an academic point of view, financial time series represent an extremely rich and fascinating source of questions, where a trace of human activity is recorded and stored in a quantitative way, sometimes over hundreds of years. What are then the statistical features of a financial time series? Does it share common signatures with other signals that physicists have learned to cope with? Once we have a good model for price changes, what is it useful for? Do we understand the basic mechanisms, in terms of human psychology, market micro-structure, etc., that are responsible for the observed statistical peculiarities of price changes, and their *universality* across markets and epochs? These are the questions that were addressed during the 2001 Altenberg lectures, and which the following notes will briefly summarize. A fuller account can be found in recently published books on the subject [1–3]. Other sources of information about this field can be found in Ref. [4], on [www.science-finance.fr](http://www.science-finance.fr) and in the recent journals *International Journal of Theoretical and Applied Finance* and *Quantitative Finance*.

## 2. ‘Stylized facts’ about financial time series

The study of price changes (actually relative price changes) on many different assets (stocks, stock indices, currencies, etc.) has revealed a number of robust features that people in the field like to call ‘stylized facts’ (see e.g. Ref. [5] for a recent review). Let us list a few of them, which are most relevant for option pricing and risk control, and that a microscopic (trader-based) model should *in fine* be able to account for:

- Relative price changes are in a good approximation uncorrelated beyond a time scale of the order of tens of minutes (on liquid markets). This means that the square of the (log) price difference grows linearly with time, with a prefactor called the *volatility*. This volatility is of the order of 3% per square-root day for individual stocks, 1% for stock indices but only 0.03% for short-term interest rates. A more detailed analysis, however, shows that some small correlations are present on the scale of a few days.
- The distribution of relative price changes  $\eta$  is strongly nonGaussian: these distributions can be characterized by Pareto (power-law) tails  $\eta^{-1-\mu}$  with an exponent  $\mu$  close to 3 for rather liquid markets [6–11]. Emerging markets have even more extreme tails, with an exponent  $\mu$  that can be less than two—in which case the volatility is infinite.
- Another striking feature is the intermittent nature of the fluctuations: localized outbursts of volatility can clearly be identified. This feature, known as *volatility clustering* [12,13,2,1], is very reminiscent of similar intermittent fluctuations in turbulent

flows [14]. This effect can be analysed more quantitatively: the temporal correlation function of the daily volatility can be fitted by an inverse power of the lag, with a rather small exponent in the range 0.1–0.3 [13,15–18]. This slow decay of the volatility correlation function leads to a multifractal-like behaviour of price changes [18–24]: the kurtosis of the (log) price difference only decays as a small power of time, rather than the inverse of time as would be the case if volatility was constant or only had short-ranged correlations [15,1,24]. This slow decay of the kurtosis has important consequences for option pricing [15,25] (see below).

- The volume of activity also shows long-ranged correlations, very similar to those observed on the volatility. This is not surprising since volatility and volume are strongly correlated [26,27].
- Past price changes and future volatilities are negatively correlated—this is the so-called ‘leverage effect’, which reflects the fact that markets become more active after a price drop, and then to calm down when the price rises. This correlation is most visible on stock indices and is characterized by a time scale of the order of 10 days [28]. This leverage effect leads to an anomalous negative skew in the distribution of price changes as a function of time, and is again important for option pricing [29].
- Inter-stock correlations also show very interesting features, such as an apparent increase of inter-stock correlations in volatile periods [30]. This is most relevant for risk control, since an increase of correlations means that risk diversification is more difficult. The full correlation matrix between all pairs of stocks can be studied, and leads to the following message: most of the eigenvalues of this matrix are well accounted for by a random matrix theory [31,32]. The dominant part of the correlation can be explained by a simple ‘one-factor’ model, where stock price changes share a common factor, called the ‘market’ [33–35].
- Interest rates corresponding to different maturities also evolve in an interesting correlated manner, which recalls the motion of an elastic string subject to noise. For recent work on this subject, see Refs. [36,37].

The most important message of these empirical studies is that prices behave very differently from the simple geometric Brownian motion which is often assumed in mathematical finance: extreme events are much more probable, and interesting nonlinear correlations (volatility–volatility and price–volatility) are observed.

### 3. Implications for option pricing

#### 3.1. General framework

We now turn to the problem of option pricing and hedging when the statistics for price increments  $\delta x$  have the nonGaussian properties discussed above. The distinctive feature of the continuous time random walk model usually considered in the theory of option pricing is the possibility of perfect hedging (for a remarkable introduction see Ref. [38]), that is, a complete elimination of the risk associated to option trading [38,39]. This property, however, no longer holds for more realistic models [1].

Let us write down the global wealth balance  $\Delta W|_0^T$  associated with the writing of a ‘call’ option on an asset of price  $x(t)$ , of maturity  $T$  and exercise price  $x_s$  [1]:

$$\Delta W|_0^T = \mathcal{C}(x_0, x_s, T) \exp(rT) - \max(x(T) - x_s, 0) + \sum_i \phi(x_i, t_i) \exp(r(T - t_i)) [x_{i+1} - x_i - rx_i \tau], \tag{1}$$

where  $\mathcal{C}(x_0, x_s, T)$  is the price of the call,  $x_0 = x(t=0)$ ,  $x_i = x(t_i)$  is the price of the asset on which the option is written, and  $\phi(x, t)$  the trading strategy, i.e., the number of stocks per option in the portfolio of the option writer. Finally,  $r$  is the (constant) interest rate, and  $\tau$  the elementary time interval. This wealth balance contains three terms:

- The first term is the price of the contract, paid at time  $t = 0$  and used to buy bonds that yield the rate  $r$ .
- The second term defines the option contract: the profit of the buyer of the option is equal to  $x_s - x(T)$  if  $x(T) > x_s$  (i.e., if the option is exercised) and zero otherwise—the option is an insurance contract which guarantees to its owner a maximum price for acquiring a certain stock at time  $T$ . Conversely, a ‘put’ option would guarantee a certain minimum price for the stock held by the owner of the option.
- The third term counts the profit or loss (over the risk-free rate) incurred due to the trading strategy  $\phi$  (see Ref. [1] for details).

A natural procedure to fix the price of the option  $\mathcal{C}(x_0, x_s, T)$  and the optimal strategy  $\phi^*(x, t)$  was proposed in Refs. [40–42] and further discussed in Ref. [1]. It consists in imposing a *fair game condition*, i.e.:

$$\langle \Delta W|_0^T[\phi] \rangle = 0 \tag{2}$$

and a *risk minimization condition*:

$$\left. \frac{\delta \langle \Delta W|_0^T[\phi]^2 \rangle}{\delta \phi(x, t)} \right|_{\phi^*} = 0. \tag{3}$$

Here, we assume that the variance of the wealth variation is a relevant measure of the risk. However, other measures are possible, such as higher moments of the distribution of  $\Delta W$ , or the ‘value-at-risk’, which is directly related to the weight contained in the negative tails of the distribution of  $\Delta W|_0^T$  [43,1].

The notation  $\langle \dots \rangle$  in Eqs. (2) and (3) means that one averages over the probability of the different trajectories. The explicit solution of Eqs. (2) and (3) for a general uncorrelated process (i.e.,  $\langle \delta x_i \delta x_j \rangle = 0$  for  $i \neq j$ , where  $\delta x_i = x_{i+1} - x_i$ ) is relatively easy to write if the average bias  $\langle \delta x_i \rangle$  and the interest rate  $r$  are negligible,<sup>1</sup> which is the case for short maturities  $T$ . In this case, one finds:

$$\mathcal{C}(x_0, x_s, T) = \int_{x_s}^{\infty} dx' (x' - x_s) P(x', T | x_0, 0), \tag{4}$$

$$\phi^*(x, t) = \int_{x_s}^{\infty} dx' \langle \delta x \rangle_{(x,t) \rightarrow (x',T)} \frac{(x' - x_s)}{\sigma^2(x, t)} P(x', T | x, t), \tag{5}$$

---

<sup>1</sup> For the general case, see Refs. [1,44–46].

where  $\sigma^2(x, t) = \langle \delta x^2 \rangle_{|x, t}$  is the ‘local volatility’—which may depend on  $x, t$ —and  $\langle \delta x \rangle_{(x, t) \rightarrow (x', T)}$  is the mean instantaneous increment conditioned to the initial condition  $(x, t)$  and a final condition  $(x', T)$ . The *minimal* residual risk, defined as  $\mathcal{R}^* = \langle \Delta W|_0^T [\phi^*]^2 \rangle$  is in general strictly positive (and in practice rather large), except for Gaussian fluctuations *in the continuous limit*, where the residual risk is strictly zero! In this limit, the above Eqs. (2) and (3) actually exactly lead to the celebrated Black–Scholes option pricing formula. In particular, one can indeed check that  $\phi^*$  is in that case related to  $\mathcal{C}$  through:  $\phi^* = \partial \mathcal{C}(x_0, x_s, T) / \partial x_0$ , which corresponds to the so-called  $\Delta$ -hedge found by Black and Scholes.

### 3.2. Cumulant expansion and volatility smile

In the case where the market fluctuations are moderately nonGaussian, one might expect that a *cumulant expansion* around the Black–Scholes formula leads to interesting results. This cumulant expansion has been worked out in general in Ref. [1] (see also Ref. [25]), both for the price and for the optimal strategy. If one only retains the leading order correction which is (for symmetric fluctuations) proportional to the kurtosis  $\kappa_T$ , one finds that the price of options  $\mathcal{C}(x_0, x_s, T)$  can be written as a Black–Scholes formula, but with a modified value of the volatility  $\sigma$ , which becomes price and maturity dependent [15]:

$$\sigma_{imp.}(x_s, T) = \sigma \left[ 1 + \frac{\kappa_T}{24} \left( \frac{(x_s - x_0)^2}{\sigma^2 T} - 1 \right) \right]. \quad (6)$$

The volatility  $\sigma_{imp.}$  is called the implied volatility by the market operators, who use the standard Black–Scholes formula to price options, but with a value of the volatility which they estimate intuitively, and which turns out to depend on the exercise price in a roughly parabolic manner, as indeed suggested by Eq. (6).

This is the so-called ‘volatility smile’. Eq. (6), furthermore, shows that the curvature of the smile is directly related to the kurtosis  $\kappa_T$  of the underlying statistical process on the scale of the maturity  $T = N\tau$ . When the price distribution is skewed, as is the case for stock indices where the leverage effect induces a significant negative skew, there are corrections to the above smile formula: the smile itself become asymmetric.

We have tested this prediction by directly comparing the ‘implied kurtosis’, obtained by extracting from real option prices (on the BUND market) the volatility  $\sigma$  (which turns out to be highly correlated with a short time filter of the historical volatility), and the curvature of the implied volatility smile, to the historical value of the kurtosis  $\kappa_T$  discussed above. We find a remarkable agreement between the implied and historical kurtosis [15,1]. This and the fact that they evolve similarly with maturity, shows that the market as a whole is able to correct (by trial and errors) the inadequacies of the Black–Scholes formula, to encode in a satisfactory way both the fact that the distribution has a positive kurtosis, and that this kurtosis decays in an anomalous fashion due to volatility persistence effects. However, the real risks associated with option trading are, at present, not satisfactorily estimated by market participants. In particular, most risk control softwares dealing with option books are based on a Gaussian description of the fluctuations.

#### 4. Simple models for herding and mimicry

We now turn to simple models for thick tails in the distribution of price increments in financial markets. An intuitive explanation is herding: if a large number of agents acting on markets coordinate their action, the price change is likely to be huge due to a large imbalance between buy and sell orders [47]. However, this coordination can result from two rather different mechanisms.

- One is the feedback of past price changes onto themselves, which we will discuss in the following section. Since all agents are influenced by the very *same* price changes, this can induce nontrivial collective behaviour: for example, an accidental price drop can trigger large sell orders, which lead to further downward moves.
- The second is direct influence between the traders, through exchange of information that leads to ‘clusters’ of agents sharing the same decision to buy, sell, or be inactive at any given instant of time.

##### 4.1. Herding and percolation

A simple model of how herding affects the price fluctuations was proposed in Ref. [48].<sup>2</sup> It assumes that the price increment  $\delta x$  depends linearly on the instantaneous offset between supply and demand [48,49]. More precisely, if each operator in the market  $i$  wants to buy or sell a certain fixed quantity of the financial asset, one has [48]:<sup>3</sup>

$$\delta x = \frac{1}{\lambda} \sum_i \varphi_i, \quad (7)$$

where  $\varphi_i$  can take the values  $-1, 0$  or  $+1$ , depending on whether the operator  $i$  is selling, inactive, or buying, and  $\lambda$  is a measure of the market depth. Note that the linearity of this relation, even for small arguments, has been questioned by Zhang [50]. Recent empirical analysis, however, seems to confirm that the relation is indeed linear for small arguments, but bends down and even saturates for larger arguments [51].

Suppose now that the operators interact among themselves in a heterogeneous manner: with a small probability  $c/N$  (where  $N$  is the total number of operators on the market), two operators  $i$  and  $j$  are ‘connected’, and with probability  $1 - c/N$ , they ignore each other. The factor  $1/N$  means that on average, the number of operators connected to any particular one is equal to  $c$  (the resulting graph is precisely the same as the random trading graph of Section 3.1). Suppose finally that if two operators are connected, they come to agree on the strategy they should follow, i.e.,  $\varphi_i = \varphi_j$ .

<sup>2</sup> See the numerous references of this paper for other works on herding in economics and finance.

<sup>3</sup> This can alternatively be written for the relative price increment  $\delta x/x$ , which is more adapted to describe long time scales. On short time scales, however, an additive model is often preferable. See the discussion in Refs. [1,28].

It is easy to understand that the population of operator clusters into groups sharing the same opinion. These clusters are defined such that there exists a connection between any two operators belonging to this cluster, although the connection can be indirect and follow a certain ‘path’ between operators. These clusters do not have all the same size, i.e., do not contain the same number of operators. If the size of cluster  $\mathcal{C}$  is called  $S(\mathcal{C})$ , one can write:

$$\delta x = \frac{1}{\lambda} \sum_{\mathcal{C}} S(\mathcal{C}) \varphi(\mathcal{C}), \quad (8)$$

where  $\varphi(\mathcal{C})$  is the common opinion of all operators belonging to  $\mathcal{C}$ . The statistics of the price increments  $\delta x$  therefore reduces to the statistics of the size of clusters, a classical problem in percolation theory [52]. One finds that as long as  $c < 1$  (less than one ‘neighbour’ on average with whom one can exchange information), then all  $S(\mathcal{C})$ ’s are small compared to the total number of traders  $N$ . More precisely, the distribution of cluster sizes takes the following form in the limit where  $1 - c = \varepsilon \ll 1$ :

$$P(S) \propto_{S \gg 1} \frac{1}{S^{5/2}} \exp - \varepsilon^2 S, \quad S \ll N. \quad (9)$$

When  $c = 1$  (percolation threshold), the distribution becomes a pure power-law with an exponent  $\mu = \frac{3}{2}$ , and the Central Limit Theorem tells us that in this case, the distribution of the price increments  $\delta x$  is precisely a pure symmetrical Lévy distribution of index  $\mu = \frac{3}{2}$  [1] (assuming that  $\varphi = \pm 1$  play identical roles, that is if there is no global bias pushing the price up or down). If  $c < 1$ , on the other hand, one finds that the Lévy distribution is truncated exponentially, leading to a larger effective tail exponent  $\mu$  [48]. If  $c > 1$ , a finite fraction of the  $N$  traders have the same opinion: this leads to a crash. This simple model has been extended in several directions by Stauffer and collaborators [53]. Very recently, a somewhat related model was studied in Ref. [54] where each agent probes the opinion of a pool of  $m$  randomly selected agents. The agent then chooses either to conform to the majority opinion or to be contrarian if the majority is too strong. This interesting model leads to various types of behaviour, including a chaotic phase.

#### 4.2. Avalanches of opinion changes

The above simple percolation model is interesting but has one major drawback: one has to assume that the parameter  $c$  is smaller than one, but relatively close to one such that Eq. (9) is valid, and nontrivial statistics follows. One should thus explain why the value of  $c$  spontaneously stabilizes in the neighbourhood of the critical value  $c = 1$ . Furthermore, this model is purely static, and does not specify how the above clusters evolve with time. As such, it cannot address the problem of volatility clustering. Several extensions of this simple model have been proposed [53,55], in particular to increase the value of  $\mu$  from  $\mu = 3/2$  to  $\mu \sim 3$  and to account for volatility clustering.

One particularly interesting model is inspired by the recent work of Dahmen and Sethna [56,57], that describes the behaviour of random magnets in a time-dependent

magnetic field. Transposed to the present problem (as first suggested in Ref. [1]), this model describes the collective behaviour of a set of traders exchanging information, but having all different a priori opinions. One trader can, however, change his mind and take the opinion of his neighbours if the coupling is strong, or if the strength of his a priori opinion is weak. More precisely, the opinion  $\varphi_i(t)$  of agent  $i$  at time  $t$  is determined as

$$\varphi_i(t) = \text{sign} \left( h_i(t) + \sum_{j=1}^N J_{ij} \varphi_j(t) \right), \tag{10}$$

where  $J_{ij}$  is a connectivity matrix describing how strongly agent  $j$  affects agent  $i$ , and  $h_i(t)$  describes the a priori opinion of agent  $i$ :  $h_i > 0$  means a propensity to buy,  $h_i < 0$  a propensity to sell. We assume that  $h_i$  is a random variable with a time-dependent mean  $\bar{h}(t)$  and root mean square  $\Delta$ . The quantity  $\bar{h}(t)$  represents for example confidence in long-term economy growth ( $\bar{h}(t) > 0$ ), or fear of recession ( $\bar{h}(t) < 0$ ), leading to a nonzero average pessimism or optimism. Suppose that one starts at  $t = 0$  from a ‘euphoric’ state, where  $\bar{h} \gg \Delta, J$ , such that  $\varphi_i = 1$  for all  $i$ ’s.<sup>4</sup> Now, confidence is decreased progressively. How will sell orders appear?

Interestingly, one finds that for small enough influence (or strong heterogeneities of agents’ anticipations), i.e., for  $J \ll \Delta$ , the average opinion  $O(t) = \sum_i \varphi_i(t)/N$  evolves continuously from  $O(t = 0) = +1$  to  $O(t \rightarrow \infty) = -1$ . The situation changes when imitation is stronger since a discontinuity then appears in  $O(t)$  around a ‘crash’ time  $t_c$ , when a finite fraction of the population simultaneously change opinion. The gap  $O(t_c^-) - O(t_c^+)$  opens continuously as  $J$  crosses a critical value  $J_c(\Delta)$  [56]. For  $J$  close to  $J_c$ , one finds that the sell orders again organize as avalanches of various sizes, distributed as a power-law with an exponential cut-off. In the ‘mean-field’ case where  $J_{ij} \equiv J/N$  for all  $ij$ , one finds  $\mu = 5/4$ . Note that in this case, the value of the exponent  $\mu$  is universal, and does not depend, for example, on the shape of the distribution of the  $h_i$ ’s, but only on some global properties of the connectivity matrix  $J_{ij}$ .

A slowly oscillating  $\bar{h}(t)$  can therefore lead to a succession of bull and bear markets, with a strongly nonGaussian, intermittent behaviour, since most of the activity is concentrated around the crash times  $t_c$ . Some modifications of this model are, however, needed to account for the empirical value  $\mu \sim 3$  observed on the distribution of price increments (see the discussion in Ref. [53]).

Note that the same model can be applied to other interesting situations, for example to describe how applause end in a concert hall (here,  $\varphi = \pm 1$  describes, respectively, a clapping and a quiet person, and  $O(t)$  is the total clapping activity). Clapping can end abruptly when imitation is strong, or smoothly when many fans are present in the audience. A static version of the same model has been proposed to describe rational group decision making [58].

---

<sup>4</sup> Here  $J$  denotes the order of magnitude of  $\sum_j J_{ij}$ .

## 5. Models of feedback effects on price fluctuations

### 5.1. Risk-aversion induced crashes

The above average ‘stimulus’  $\bar{h}(t)$  may also strongly depend on the past behaviour of the price itself. For example, past positive trends are, for many investors, incentives to buy, and vice versa. Actually, for a given trend amplitude, price drops tend to feed back more strongly on investors’ behaviour than price rises. Risk-aversion creates an asymmetry between positive and negative price changes [59]. This is reflected by option markets, where the price of out-of-the-money puts (i.e., insurance against crashes) is anomalously high.

Similarly, past periods of high volatility increases the risk of investing in stocks, and usual portfolio theories then suggest that sell orders should follow. A simple mathematical transcription of these effects is to write Eq. (7) in a linearized, continuous time form:<sup>5</sup>

$$\frac{dx}{dt} \equiv u = \frac{1}{\lambda} \bar{h}(t), \quad (11)$$

and write a dynamical equation for  $\bar{h}(t)$  which encodes the above feedback effects [59,49]:

$$\frac{d\bar{h}}{dt} = a\bar{h} - b\bar{h}^2 - c\bar{h} + \eta(t), \quad (12)$$

where  $a$  describes trends following effects,  $b$  risk aversion effects (breaking the  $u \rightarrow -u$  symmetry),  $c$  is a mean reverting term which arises from market clearing mechanisms (the very fact that the price moves clears a certain number of orders), and  $\eta$  is a noise term representing random external news [59]. Eliminating  $\bar{h}$  from the above equations leads to

$$\frac{du}{dt} = -\gamma u - \beta u^2 + \frac{1}{\lambda} \eta(t) \equiv -\frac{\partial V}{\partial u} + \frac{1}{\lambda} \eta(t), \quad (13)$$

where  $\gamma = (c - a)/\lambda$  and  $\beta = b/\lambda$ . Eq. (13) represents the evolution of the position  $u$  of a viscous fictitious particle in a ‘potential’  $V(u) = \gamma u^2/2 + \beta u^3/3$ . If trend following effects are not too strong,  $\gamma$  is positive and  $V(u)$  has a local minimum for  $u = 0$ , and a local maximum for  $u^* = -\gamma/\beta$ , beyond which the potential plummets to  $-\infty$ .<sup>6</sup> A ‘potential barrier’  $V^* = \gamma u^{*2}/6$  separates the (meta-)stable region around  $u = 0$  from the unstable region. The nature of the motion of  $u$  in such a potential is the following: starting at  $u = 0$ , the particle has a random harmonic-like motion in the vicinity of  $u = 0$  until an ‘activated’ event (i.e., driven by the noise term) brings the particle near  $u^*$ . Once this barrier is crossed, the fictitious particle reaches  $-\infty$  in finite time. In financial

<sup>5</sup> In the following, the herding effects described by  $J_{ij}$  are neglected, or more precisely, only their *average effect* encoded by  $\bar{h}$  is taken into account.

<sup>6</sup> If  $\gamma$  is negative, the minimum appears for a positive value of the return  $u^*$ . This corresponds to a speculative bubble. See Ref. [59].

terms, the regime where  $u$  oscillates around  $u = 0$  and where  $\beta$  can be neglected, is the ‘normal’ fluctuation regime. This normal regime can, however, be interrupted by ‘crashes’, where the time derivative of the price becomes very large and negative, due to the risk aversion term  $b$  which destabilizes the price by amplifying the sell orders. The interesting point is that these two regimes can be clearly separated since the average time  $t^*$  needed for such crashes to occur is exponentially large in  $V^*$  [60], and can thus appear only very rarely. A very long time scale is therefore naturally generated in this model.

Note that in this line of thought, a crash occurs because of an improbable succession of unfavourable events, and not due to a single large event in particular. Furthermore, there are no ‘precursors’ in this model: before  $u$  has reached  $u^*$ , it is impossible to decide whether it will do so or whether it will quietly come back in the ‘normal’ region  $u \simeq 0$ . Solving the Fokker–Planck equation associated to the Langevin equation (13) leads to a stationary state with a power law tail for the distribution of  $u$  (i.e., of the instantaneous price increment) decaying as  $u^{-2}$  for  $u \rightarrow -\infty$ . More generally, if the risk aversion term took the form  $-b|u|^{1+\mu}$ , the negative tail would decay as  $u^{-1-\mu}$ .

### 5.2. Dynamical volatility models

The simplest model that describes volatility feedback effects is to write an ARCH-like equation [61], which relates today’s activity to a measure of yesterday’s activity, for example:

$$\sigma_k = \sigma_{k-1} + K(\sigma_0 - \sigma_{k-1}) + g|\delta x_{k-1}|, \tag{14}$$

where  $\sigma_0$  is an average volatility level,  $K$  a mean-reverting term, and  $g$  describes how much yesterday’s observed price change affects the behaviour of traders today. Since  $|\delta x_{k-1}|$  is a noisy version of  $\sigma_{k-1}$ , the above equation is, in the continuous time limit, a Langevin equation with *multiplicative* noise:

$$\frac{d\sigma}{dt} = K(\sigma_0 - \sigma) + g\sigma\eta(t). \tag{15}$$

The stationary distribution corresponding to this equation is

$$P_{eq}(\sigma) = \frac{(\mu - 1)^\mu}{\Gamma[\mu]} \frac{e^{-(\mu-1)/\sigma}}{\sigma^{1+\mu}}, \tag{16}$$

where the tail exponent is by  $\mu - 1 \propto K/g^2$ : over-reactions to past informations (i.e., large values of  $g$ ) decreases the tail exponent  $\mu$ . Therefore, one finds a nonuniversal exponent in this model, which is bequeathed to the distribution of price increments if one assumes that the volatility fluctuations are the dominant cause of large changes.

Note that the temporal correlation function of the volatility  $\sigma$  can be exactly calculated within this model [62], and is found to be exponentially decaying, at variance with the slow power-law (or logarithmic) decay observed empirically. Furthermore,

distribution (16) does not concur with the nearly log-normal distribution of the volatility that seems to hold empirically [16,67].

At this point, the slow decay of the volatility can be ascribed to two rather different mechanisms. One is the existence of traders with many different time horizons, as suggested in Refs. [63,15]. If traders are affected not only by yesterday's price change amplitude  $|\delta x_{k-1}|$ , but also by price changes on coarser time scales  $|x_k - x_{k-p}|$ , then the correlation function is expected to be a sum of exponentials with decay rates given by  $p^{-1}$ . Interestingly, if the different  $p$ 's are uniformly distributed on a log scale, the resulting sum of exponentials is to a good approximation decaying as a logarithm, as advocated in Ref. [18]. More precisely:

$$C(\tau) = \frac{1}{\log(p_{\max}/p_{\min})} \int_{p_{\min}}^{p_{\max}} d(\log p) \exp(-\tau/p) \simeq \frac{\log(p_{\max}/\tau)}{\log(p_{\max}/p_{\min})}, \quad (17)$$

whenever  $p_{\min} \ll \tau \ll p_{\max}$ . Now, the human time scales are indeed in a natural pseudo-geometric progression: hour, day, week, month, trimester, year [15].

Yet, some recent numerical simulations of traders allowed to switch between different strategies (active/inactive, or chartist/fundamentalist) suggest strongly intermittent behaviour [55,64–66,68], and a slow decay of the volatility correlation function without the explicit existence of logarithmically distributed time scales. Is there a simple, universal mechanism that could explain these ubiquitous long-range volatility correlations?

A possibility, discussed in Ref. [69], is that the volume of activity exhibits long-range correlations because agents switch between different strategies depending on their relative performance. Imagine for example that each agent has two strategies, one active strategy (say trading every day), and one inactive, or less active strategy. The 'score' of the inactive strategy (i.e., its cumulative profit) is constant in time, or more precisely equal to the long-term average growth rate. The score of active strategy, on the other hand, fluctuates up and down, due to the fluctuations of the market prices themselves. Since to a good approximation the market prices are not predictable, this means that the score of any active strategy will behave like a random walk, with an average equal to that of the inactive strategy (assuming that transaction costs are small). Therefore, on some occasions the score of the active strategy will happen to be higher than that of the inactive strategy and the agent will be active, before the score of the active strategy crosses that of the inactive strategy. The time during which an agent is active is thus a random variable with the same statistics as the return time to the origin of a random walk (the difference of the scores of the two strategies). Interestingly, the return times of a random walk are well known to be very broadly distributed: the *average* return time is actually infinite. Hence, if one computes the correlation of activity in such a model, one finds long-range correlations due to long periods of times where many agents are active (or inactive). One can consider a specific model (the 'Grand Canonical Minority Game'; for papers on minority game see e.g. Ref. [70]) where this scenario can be studied more quantitatively, and have found that indeed long-range correlations in the volume of activity are observed: see Fig. 1. More precisely, the

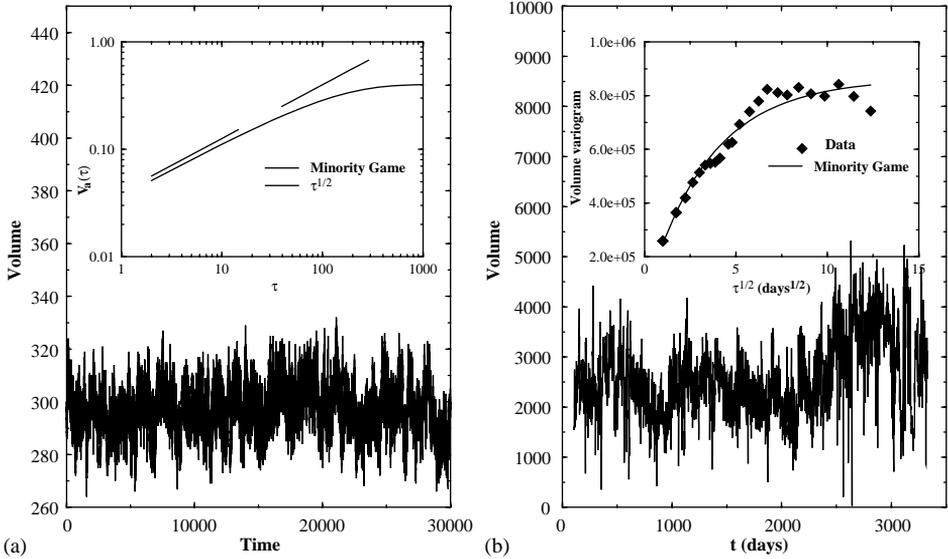


Fig. 1. (a) Volume of activity  $v(t)$  (number of active agents) as a function of time for the Grand Canonical Minority Game (GCMG). Inset: The corresponding activity variogram  $\langle(v(t + \tau) - v(t))^2\rangle$ , as a function of the lag  $\tau$ , in a log–log plot to emphasize the predicted  $\sqrt{\tau}$  singularity at small  $\tau$ 's. (b) Total daily volume of activity (number of trades) on the S&P 500 future contracts in the years 1985–1998. Inset: Corresponding variogram (diamonds) as a function of the square root of the lag. Note the clear linear behaviour for small  $\sqrt{\tau}$ . The full line is the GCMG fit, with both axes rescaled and a constant added to account for the presence of ‘white noise’ trading.

variogram of the volume of activity, defined as  $V(\tau) = \langle(v(t + \tau) - v(t))^2\rangle_t$ , can be very accurately fitted by

$$V(\tau)|_{fit} = V^\infty \left( 1 - \exp\left(-\sqrt{\frac{\tau}{\tau_0}}\right) \right). \tag{18}$$

This simple model even allows one to reproduce quantitatively the volume of activity correlations observed on the New York stock exchange market: see Fig. 1. This mechanism is very generic and probably also explains why this effect arises in more realistic market models [64,68].

As shown in Ref. [69], this mechanism can thus explain the long-range volatility correlations observed on all financial markets. However, this interpretation is quite different from the ‘cascade’ picture proposed in Refs. [19,21,18], where the volatility results from some sort of multiplicative random process (which actually naturally leads to the log-normal volatility distribution actually observed empirically). More work is needed to establish which of these two mechanisms is most relevant in financial markets.

## Acknowledgements

I would like to thank in particular P. Cizeau, R. Cont, I. Giardina, P. Gopikrishnan, L. Laloux, T. Lux, A. Matacz, M. Mézard, and M. Potters for many discussions on these matters.

## References

- [1] J.-P. Bouchaud, M. Potters, *Théorie des Risques Financiers*, Aléa-Saclay, 1997; *Theory of Financial Risks*, Cambridge University Press, Cambridge, 2000.
- [2] R. Mantegna, H.E. Stanley, *An Introduction to Econophysics*, Cambridge University Press, Cambridge, 1999.
- [3] J. Voit, *The Statistical Mechanics of Financial Markets*, Springer, Berlin, 2001.
- [4] J.D. Farmer, Physicists attempt to scale the ivory towers of finance, *Int. J. Theor. Appl. Finance* 3 (2000) 311.
- [5] R. Cont, *Quantitative Finance* 1 (2001) 223.
- [6] V. Plerou, P. Gopikrishnan, L.A. Amaral, M. Meyer, H.E. Stanley, *Phys. Rev. E* 60 (1999) 6519.
- [7] T. Lux, *Appl. Financial Econom.* 6 (1996) 463.
- [8] M.M. Dacorogna, U.A. Müller, R.J. Nagler, R.B. Olsen, O.V. Pictet, *J. Int. Money Finance* 12 (1993) 413.
- [9] D.M. Guillaume, M.M. Dacorogna, R.D. Davé, U.A. Müller, R.B. Olsen, O.V. Pictet, *Finance Stochastics* 1 (1997) 95.
- [10] F. Longin, *J. Business* 69 (1996) 383.
- [11] J. Nuyts, I. Platten, Phenomenology of the term structure of interest rates with Padé approximants, e-print cond-mat/9901096.
- [12] A. Lo, *Econometrica* 59 (1991) 1279.
- [13] Z. Ding, C.W.J. Granger, R.F. Engle, *J. Empirical Finance* 1 (1993) 83.
- [14] U. Frisch, *Turbulence: The Legacy of A. Kolmogorov*, Cambridge University Press, Cambridge, 1997.
- [15] M. Potters, R. Cont, J.-P. Bouchaud, *Europhys. Lett.* 41 (1998) 239.
- [16] Y. Liu, P. Cizeau, M. Meyer, C.-K. Peng, H.E. Stanley, *Physica A* 245 (1997) 437.
- [17] P. Cizeau, Y. Liu, M. Meyer, C.-K. Peng, H.E. Stanley, *Physica A* 245 (1997) 441.
- [18] J.-F. Muzy, J. Delour, E. Bacry, *Eur. Phys. J. B* 17 (2000) 537–548; *Physica A* 299 (2001) 284.
- [19] The first paper to suggest this analogy is S. Ghashghaie, W. Breymann, J. Peinke, P. Talkner, Y. Dodge, *Nature* 381 (1996) 767.
- [20] A. Arnéodo, J.-F. Muzy, D. Sornette, *Eur. Phys. J. B* 2 (1998) 277.
- [21] A. Fisher, L. Calvet, B.B. Mandelbrot, Multifractality of DEM/\$ rates, Cowles Foundation Discussion Paper 1165; B.B. Mandelbrot, *Sci. Am.* Feb. (1999).
- [22] F. Schmitt, D. Schertzer, S. Lovejoy, *Appl. Stochastic Models Data Anal.* 15 (1999) 29.
- [23] M.-E. Brachet, E. Taflin, J.M. Tchéou, Scaling transformation and probability distributions for financial time series, e-print cond-mat/9905169.
- [24] J.P. Bouchaud, M. Potters, M. Meyer, *Eur. Phys. J. B* 13 (1999) 595.
- [25] D. Backus, S. Foresi, K. Lai, L. Wu, Accounting for biases in Black–Scholes, Working paper, 1997.
- [26] G. Bonnano, F. Lillo, R. Mantegna, Dynamics of the number of trades in financial securities, e-print cond-mat/9912006.
- [27] V. Plerou, P. Gopikrishnan, L.A. Amaral, X. Gabaix, H.E. Stanley, e-print cond-mat/9912051.
- [28] J.P. Bouchaud, A. Matacz, M. Potters, The leverage effect in financial markets: retarded volatility and market panic, e-print cond-mat/0101120, *Phys. Rev. Lett.* 87 (2001) 228701.
- [29] J.P. Fouque, G. Papanicolaou, R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press, Cambridge, 2000.
- [30] F. Longin, B. Solnik, Correlation structure of international equity markets during extremely volatile periods, Working paper, 1999.
- [31] L. Laloux, P. Cizeau, J.P. Bouchaud, M. Potters, *Phys. Rev. Lett.* 83 (1999) 1467.

- [32] V. Plerou, P. Gopikrishnan, B. Rosenow, L. Nunes Amaral, H. Stanley, *Phys. Rev. Lett.* 83 (1999) 1471.
- [33] P. Cizeau, M. Potters, J.-P. Bouchaud, Correlation structure of extreme stock returns, *Quantitative Finance* 1 (2001) 217.
- [34] F. Lillo, R.N. Mantegna, Variety and volatility in financial markets, *Phys. Rev. E* 62 (2000) 6126.
- [35] F. Lillo, R.N. Mantegna, Symmetry alteration of ensemble return distribution in crash and rally days of financial market, *Eur. Phys. J. B* 15 (2001) 603.
- [36] J.P. Bouchaud, N. Sagna, R. Cont, N. ElKaroui, M. Potters, Phenomenology of the interest rate curve, *Appl. Math. Finance* 6 (1999) 209.
- [37] A. Matacz, J.P. Bouchaud, *Int. J. Theor. Appl. Finance* 3 (2000) 703;  
B.E. Baaquie, S. Marakani, An empirical investigation of a Quantum Field Theory of Forward Rates, e-print cond-mat/0106317.
- [38] J.C. Hull, *Futures, Options and Other Derivative Securities*, Prentice-Hall, Englewood Cliffs, NJ, 1997.
- [39] P. Wilmott, *Derivatives, The Theory and Practice of Financial Engineering*, Wiley, New York, 1998.
- [40] J.P. Bouchaud, D. Sornette, *J. Phys. I France* 4 (1994) 863.
- [41] M. Schweizer, *Ann. Probab.* 22 (1994) 1536.
- [42] M. Schäl, *Math. Oper. Res.* 19 (1994) 121.
- [43] F. Selmi, J.P. Bouchaud, Hedging large risks reduces transaction costs, cond-mat/0005148, *The Wilmott Magazine*, [www.wilmott.com](http://www.wilmott.com).
- [44] E. Aurell, S. Simdyankin, *Int. J. Theor. Appl. Finance* 1 (1998) 1.
- [45] O. Hammarlid, *Int. J. Theor. Appl. Finance* 1 (1998) 227.
- [46] G. Wolczyńska, *Int. J. Theor. Appl. Finance* 1 (1998) 283.
- [47] For an introduction and references, see the very interesting book of R. Schiller, *Irrational Exuberance*, Princeton University Press, Princeton, NJ, 2000.
- [48] R. Cont, J.P. Bouchaud, *Macrocon. Dyn.* 4 (2000) 170.
- [49] J.D. Farmer, Market force, ecology and evolution, e-print adap-org/9812005.
- [50] Y.C. Zhang, *Physica A* 269 (1999) 30.
- [51] V. Plerou, P. Gopikrishnan, X. Gabaix, H. E. Stanley, Quantifying stock price response to demand fluctuations, e-print cond-mat/0106657.
- [52] D. Stauffer, A. Aharony, *Introduction to Percolation*, Taylor & Francis, London, 1994.
- [53] D. Sornette, D. Stauffer, H. Takayasu, Market fluctuations, multiplicative and percolation models, size effects and predictions, e-print cond-mat/9909439, and references therein.
- [54] A. Corcos, J.-P. Eckmann, A. Malaspina, Y. Malevergne, D. Sornette, Imitation and contrarian behavior: hyperbolic bubbles, crashes and chaos, e-print cond-mat/0109410.
- [55] G. Iori, A microsimulation of traders activity in the stock market: the role of heterogeneity, agents' interactions and trade frictions, e-print adap-org/9905005.
- [56] K. Dahmen, J.P. Sethna, *Phys. Rev. B* 53 (1996) 14872.
- [57] J.P. Sethna, K. Dahmen, C. Myers, *Nature* 410 (2001) 242.
- [58] S. Galam, *Physica A* 238 (1997) 66.
- [59] J.-P. Bouchaud, R. Cont, *Eur. J. Phys. B* 6 (1998) 543.
- [60] S. Chandrasekar, *Rev. Mod. Phys.* 15 (1943) 1, reprinted in *Selected papers on noise and stochastic processes*, Dover, New York, 1954.
- [61] C. Gourieroux, A. Montfort, *Statistics and Econometric Models*, Cambridge University Press, Cambridge, 1996.
- [62] R. Graham, A. Schenzle, *Phys. Rev. A* 25 (1982) 1731.
- [63] U.A. Muller, M.M. Dacorogna, R. Davé, R.B. Olsen, O.V. Pictet, *J. Empirical Finance* 4 (1997) 213.
- [64] T. Lux, M. Marchesi, *Nature* 397 (1999) 498;  
T. Lux, M. Marchesi, *Int. J. Theor. Appl. Fin.* 3 (2000) 675.
- [65] P. Bak, M. Paczuski, M. Shubik, *Physica A* 246 (1997) 430.
- [66] C. Hommes, *Quantitative Finance* 1 (2001) 149.
- [67] S. Micciché, G. Bonanno, F. Lillo, R.N. Mantegna, e-print cond-mat/0202527.
- [68] I. Giardina, J.P. Bouchaud, M. Mézard, *Physica A* 299 (2001) 28;  
I. Giardina, J.P. Bouchaud, e-print cond-mat/0206222.
- [69] I. Giardina, J.P. Bouchaud, M. Mézard, *Quantitative Finance* 1 (2001) 212.
- [70] D. Challet, A. Chessa, M. Marsili, Y.C. Zhang, *Quantitative Finance* 1 (2001) 168 and refs. therein.