Population dynamics in a random environment

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We investigate the competition between barrier slowing down and proliferation induced superdiffusion in a model of population dynamics in a random force field. A one-loop RG analysis close to the critical dimension $d_c = 2$ predicts a second order phase transition between a subdiffusive regime and a superdiffusive regime, at variance with our numerical results in $d = 1$ which suggest that a new *stable* mixed fixed point appears. We introduce the idea of proliferation assisted barrier crossing and give a Flory like argument to understand qualitatively the observed diffusive behaviour at this mixed fixed point.

The presence of disorder often radically changes the statistical properties of random walks. For example, random walks in a random potential are trapped in deep potential wells: this may lead to *sub-diffusion*, i.e. the fact that the typical distance traveled by the walkers grows more slowly than the square-root of time [1]. A much studied model exhibiting this type of behaviour is the Sinai model, where particles diffuse in a random force field in one dimension [2–4]. In this case, the energy barriers typically grow as the square root of the distance, which leads to a logarithmically slow progression of the random walkers. There are also several mechanisms which lead to *super-diffusion*. For example, if the random force field is rotational, the random walkers can be convected far away by long streamlines [1]. Another interesting mechanism of superdiffusion is *random proliferation*: suppose that each random walker can either die or give birth to new random walkers at a rate which is random, both in time and space. There is in this case a possibility for an 'outlier' random walker, that has by chance traveled a distance much greater than the square-root of time, to have been particularly prolific: he and his siblings then represent an appreciable fraction of the whole population, leading to a motion of the centre of mass faster than diffusive. This mechanism has been much studied (although not explicitely discussed as such) in the context of Directed Polymers (DP) in random media or equivalently the Kardar-Parisi-Zhang (KPZ) model of surface growth [5]. The aim of the present work is to investigate the case where both these mechanisms are present simultaneously. The motivations for such a mixed model are numerous. In the context of population dynamics (for example, bacteria on a random substrate), similar models have recently been investigated, with quite interesting results [6]. One can also give an economic interpretation of population dynamics, where the local density of random walkers is the wealth of a given individual. Biased diffusion represents trading between individuals, whereas the random growth term is the result of speculation [7]. One can argue that generically, this type of model leads to a Pareto (power-law) tail in the distribution of wealth [7]. Finally, from a theoretical point of view, this mixed model leads to the interesting possibility of a phase transition between superdiffusion and subdiffusion, with a non trivial critical behaviour.

More precisely, we study here the following equation for the local population density $P(\vec{x},t)$ in d dimensions:

$$\frac{\partial P(\vec{x},t)}{\partial t} = \nu_0 \Delta P(\vec{x},t) - \vec{\nabla}(\vec{F}(\vec{x})P) + \eta(\vec{x},t)P(\vec{x},t), \tag{1}$$

where ν_0 is the bare diffusion constant, $\vec{F}(\vec{x})$ a space dependent static Gaussian random force such that $\langle F_\mu(\vec{x})F_\nu(\vec{x}')\rangle_F = 2\sigma_F^2 \delta_{\mu,\nu}\delta^d(\vec{x}-\vec{x}')$ [8], and $\eta(\vec{x},t)$ a Gaussian random growth rate, depending both on space and time, with $\langle \eta(\vec{x},t)\eta(\vec{x}',t')\rangle_\eta = 2\sigma_\eta^2 \delta(t-t')\delta^d(\vec{x}-\vec{x}')$. Due to the last term, the total population $Z(t) = \int d\vec{x} P(\vec{x},t)$ is not conserved. The quantities of interest, which describe how the population spreads in time are, for example, the average centre of mass motion,

$$x_{cm}^2(t) = \langle \left(\frac{1}{Z} \int \vec{x} P(\vec{x},t)\, d\vec{x} \right)^2 \rangle_{F,\eta} \tag{2}$$

or the average 'width' of the diffusing packet $\langle \Delta^2 \rangle_{F,\eta}$:

$$\Delta^2(t) = \frac{1}{Z}\int \vec{x}^2 P(\vec{x},t)\, d\vec{x} - \left(\frac{1}{Z}\int \vec{x} P(\vec{x},t)\, d\vec{x} \right)^2 \tag{3}$$

(Other moments can however also be studied: see below). An alternative description is in terms of the free-energy $h(\vec{x},t) = \log P(\vec{x},t)$, which obeys the equation:

$$\frac{\partial h(\vec{x},t)}{\partial t} = \nu_0 \Delta h(\vec{x},t) + \lambda(\vec{\nabla} h)^2 - \vec{F}(\vec{x}) \cdot \vec{\nabla} h - \vec{\nabla} \cdot \vec{F}(\vec{x}) + \eta(\vec{x},t), \quad (4)$$

with $\lambda = \nu_0$. When $\vec{F} \equiv 0$, these equations represent the well-known KPZ (or Directed Polymer) problem, whereas for $\eta \equiv 0$, one recovers the problem of a random walk in a random environment. Both problems can be approached using a perturbative renormalisation group; interestingly, the critical dimension for both problems is $d_c = 2$. For the random drift problem, one finds that the coupling constant $g_F = \sigma_F^2/(2\pi)\nu_0^2$ flows towards a non trivial fixed point of order ϵ in dimensions $d = 2 - \epsilon$ [9,10,1]. This in turn leads to a subdiffusive behaviour: $x_{cm}(t)$ grows as t_F^{ν} with $\nu_F = (1 - \epsilon^2)/2 < 1/2$.

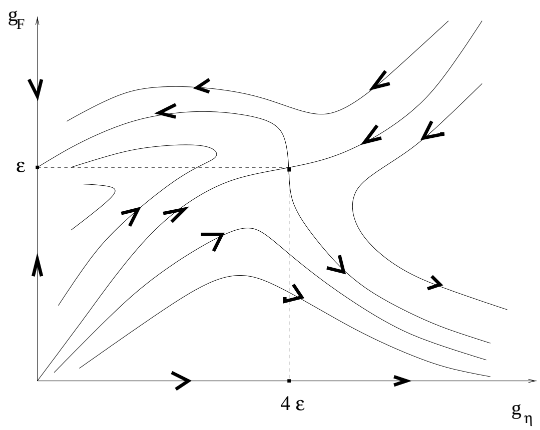

FIG. 1. One loop RG flow in the g_η, g_F plane.

For the KPZ problem, the coupling constant is $g_\eta = \sigma_\eta^2 \lambda^2/(2\pi)\nu_0^3$; the Gaussian fixed point $g_\eta = 0$ is again unstable for $d < 2$, but there is no accessible fixed point at one loop for $d > 3/2$ [11]. The exponent ν therefore cannot be computed but is expected (and found numerically) to be greater than $1/2$: in the population dynamics language, the possibility of far-away proliferation leads to superdiffusion. On the other hand, for $d < 3/2$, a physical fixed point appears; in $d = 1$, the one loop calculation even provides the exact result $\nu_\eta = 2/3$ for reasons detailed in [11]. We have performed a RG analysis in the *mixed* case where both g_F and g_η are non zero. This can be done using a field theoretical representation (Martin-Siggia-Rose) representation of Eq. (4), which allows one to generate the perturbation expansion in g_F and g_η. Performing calculations along the lines of [9–11], we find that the two β functions are given by:

$$\frac{dg_\eta}{d\ell} = \epsilon g_\eta - 2g_\eta g_F + \frac{g_\eta^2}{4}; \frac{dg_F}{d\ell} = \epsilon g_F - \frac{\epsilon g_\eta g_F}{4} - g_F^2, \quad (5)$$

where ℓ is the logarithm of the running length scale. The resulting flow is represented in Fig. 1. Interestingly, one finds a non trivial fixed point of order ϵ in the physical region, given by $g_F^* = \epsilon$, $g_\eta^* = 4\epsilon$, with one attractive and one repulsive direction. This means that for $g_F^0 > 0$, one expects a second order phase transition from a subdiffusive behaviour for small initial values of g_η^0 to a superdiffusive (KPZ or DP) behaviour for larger g_η^0. The behaviour at the transition is found to be subdiffusive, with $1/2 > \nu^* = (1 - 3\epsilon^2/4)/2 > \nu_F$. Using the linearized mapping close to the transition point, one finds that the critical behaviour $x_{cm}(t) \propto t^{\nu^*}$ holds for $x_{cm} \ll \xi \propto (\delta g)^{-1/\epsilon}$, crossing over to either subdiffusive or superdiffusive behaviour at larger distances.

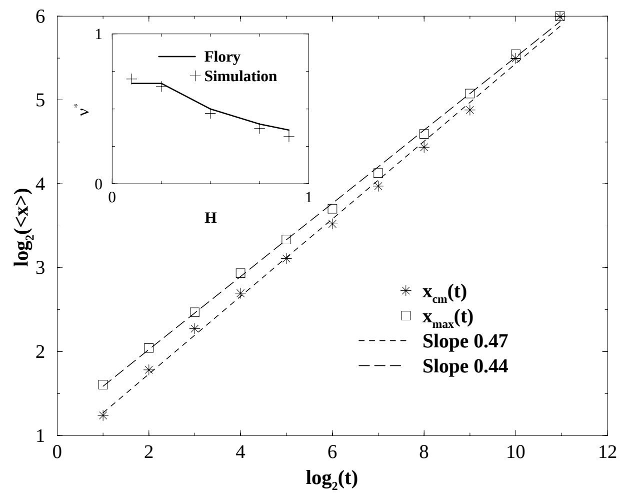

FIG. 2. Behaviour of the average centre of mass $x_{cm}(t)$ and of the average position of the maximum of the packet $x_{max}(t)$ (i.e. the point where $P(x,t)$ is maximum), as a function of time, for $\sigma_\eta/\sigma_F = 0.125$. The best linear fits are shown, and lead to the an estimate for ν^* slightly smaller than 1/2. Inset: Value of ν^* (determined from the behaviour of $x_{cm}(t)$) as a function of the Hurst exponent of the potential H, compared with the Flory prediction.

The above dynamical phase transition arises from a one loop RG analysis close to the critical dimension $d_c = 2$, and one can wonder if this picture is at least qualitatively correct in one dimension, where both the fixed points corresponding to Sinai subdiffusion and to DP/KPZ superdiffusion are well understood. We have performed numerical simulations of the mixed case in one dimension, and have found rather discordant results, which we now present and attempt to rationalize with some heuristic, Flory-like arguments. We have numerically evolved a space and time discretized version of Eq. (1), and have worked with $\log P$ to avoid precision problems. Starting from a localized packet $P(x = ia, t = 0) = \delta_{i,0}$, we have found that as soon as both coupling constants g_F^0 and g_η^0 are non zero, the exponent ν describing the diffusion of the centre of mass $x_{cm}(t)$ at large times is found to be close to the value $\nu^* = 1/2$ (see Fig. 2). The position $x_{max}(t)$ of the maximum of $P(x,t)$ behaves very similarly.

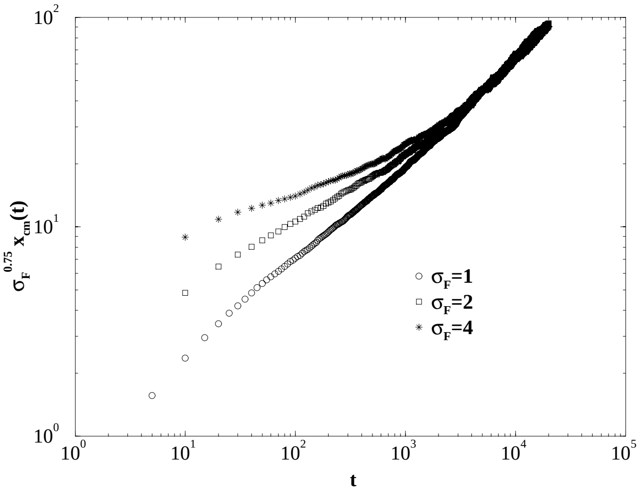

FIG. 3. Behaviour of the average centre of mass $x_{cm}(t)$ for different values of σ_F. This curves shows that for large σ_F, the short time behaviour is Sinai like, crossing over to the mixed behaviour at large times. The value of $x_{cm}(t)$ has been rescaled by $\sigma_F^{3/4}$ to obtain a reasonable data collapse at large times.

The ratio g_F^0/g_η^0 affects only the short time transient behaviour, which is either Sinai like or DP/KPZ like, as shown in Fig. 3. In other words, the non trivial fixed point g_F^*, g_η^* is found to be *attractive* in both directions, in contrast with the RG prediction. Although the value of $\nu^* = 1/2$ corresponds to free-diffusion, the motion of the packet for a given environment is very far from simple diffusion, as the study of the width Δ of the packet shows. We have found numerically that $\langle \Delta^q \rangle_{F,\eta}$ behaves as $t^{q\zeta_q}$ with $\zeta_1 \simeq 0.24$, $\zeta_2 \simeq 0.34$ and $\zeta_4 \simeq 0.38$. This non trivial behaviour is actually present for both the Sinai problem and the DP/KPZ problem. This comes from the fact that for both problems, the effective free energy $h(x,t) = \log P(x,t)$ behaves as a random walk in x space. This is trivial for the Sinai problem, since the potential is indeed constructed as the sum of local random forces. For the DP/KPZ problem, this is far less trivial and results from the fact that one can obtain exactly the stationary distribution of $h(x,t)$ in one dimension, which turns out to be the same as for the linear case $\lambda = 0$, i.e., again a random walk in x space [5]. It is well known that for a random walk potential, the probability that two nearly degenerate minima are separated by a distance Δ falls off as $\Delta^{-3/2}$ for large Δ. The qth moment of Δ is therefore dominated by extreme events as soon as $q > 1/2$. Physically, this means that for most realisations of the disorder, the width Δ of the packet is small [12,13], except in rare situations where the packet is divided into two subpackets very distant from one another. The natural cut-off for Δ is of the order of $x_{cm}(t)$ itself. Therefore one obtains, for $q > 1/2$, $\langle \Delta^q(t) \rangle \propto [x_{cm}(t)]^{q-1/2}$. For the Sinai problem, using $x_{cm}(t) \propto \log^2(t)$, this leads to $\langle \Delta^2(t) \rangle_F \propto \log^3(t)$, whereas for the DP/KPZ case, using $x_{cm}(t) \propto t^{2/3}$, one finds $\langle \Delta^2(t) \rangle_\eta \propto t$: both these results are actually exact, as has been shown in [4,13,14]. Assuming that the effective potential in the mixed case is again a random walk in x space, and using $\nu^* \simeq 1/2$, we obtain $\zeta_q = (2q-1)/4q$, i.e. $\zeta_1 \simeq 0.25$, $\zeta_2 \simeq 0.375$ and $\zeta_4 \simeq 0.4375$, in reasonable agreement with our numerical values [15].

In order to understand the value of $\nu^* \simeq 1/2$, one needs to develop a consistent picture of the competition between the slowing down induced by the ever-growing Sinai barriers and the speeding up of the population spreading allowed by the multiplicative growth term η. Before addressing the full Sinai+KPZ problem, we first consider the simpler case of a unique barrier of height U_0, which develops on scale L. For definiteness, we have solved numerically the equation (1) on the interval $[0, L]$ with $F(x) = -U_0/L \sin(4\pi x/L)$. The initial condition is localized in the first well, and the crossing time τ is defined as the average time after which the relative weight of the population in the second well is half of that in the first well. For $\eta \equiv 0$, one finds the classical Arrhenius law: $\log \tau = U_0/\nu_0$. When $\eta \neq 0$, the behaviour of τ as a function of U_0 for different values of L is shown in Fig. 4. The result can be expressed as: $\tau \propto L^{3/2} f(U_0/\sqrt{L})$, with $f(y \to 0) = 1$ and $f(y \to \infty) \propto y^b$. This scaling of τ with L can easily be understood. In the limit $U_0 \to 0$, the time for the particles to reach a distance L is given by the DP/KPZ scaling, i.e. $L \propto \tau^{2/3}$.

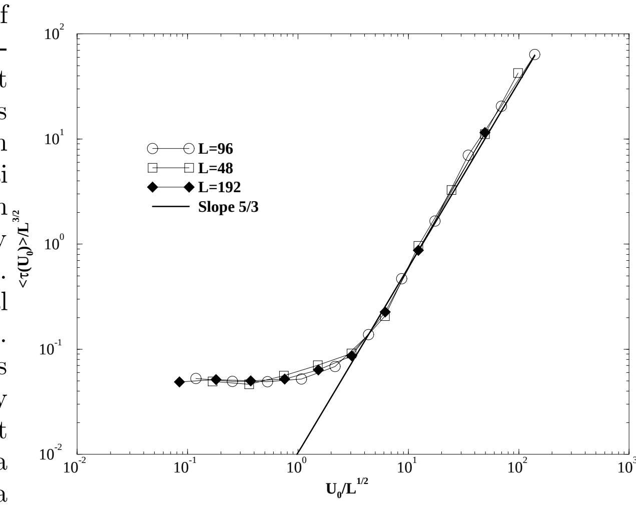

FIG. 4. Average barrier crossing time $\langle \tau \rangle$, rescaled by $L^{3/2}$, as a function of the barrier height U_0 rescaled by $\sqrt{L}$, for different sizes L, and in log-log coordinates. The slope 5/3 is shown for comparison. The power law increase of $\langle \tau \rangle$ as a function of U_0 has to be compared to the usual exponential (activated) dependence.

The influence of the external potential U_0 becomes substantial when it becomes of the order of the effective KPZ potential h, which, as discussed above, grows as $\sqrt{L}$. Numerically, the exponent b is found to be very close to $b = 5/3$. Therefore, the exponential increase of the crossing time with the barrier heigth is replaced by a power-law increase in the presence of the random growth term η. One can call this effect proliferation assisted barrier crossing: the probability that a particle reaches the top of the barrier x^* by pure diffusion is $\exp(-U_0/\nu_0)$; but due to the random growth term, this probability is multiplied by a certain proliferation 'gain' factor $\exp \mathcal{G}(x^*, t)$ [16]. If the path $\mathcal{C}$ leading from the initial point x_0 to x^* was unique, one would simply have $\mathcal{G}(x^*, t) = \int dt' \eta(x_{\mathcal{C}}(t'), t')$, which typically behaves as $\sigma_\eta \sqrt{t}$. In fact, many paths contribute to $\mathcal{G}(x^*, t)$. This leads to a kind of preaveraging effect of the random growth term η over the width $w(t)$ of the paths $\mathcal{C}$. Therefore:

$$\mathcal{G}(x^*, t) \sim \sigma_\eta \left(\int_0^t \frac{dt'}{w(t')^d} \right)^{1/2}. \tag{6}$$

Since most paths leading to x^* spend their time in the thermally accessible region of the well, one can estimate $w(t')$ as $w = L/\sqrt{U_0}$. The proliferation factor then compensates the barrier when $\tau \propto U_0^{3/2}$ (for $d = 1$). This simple argument therefore leads to $b = 3/2$, not very far from the numerical value $b \simeq 5/3$. Actually, one can apply this argument to the unconfined case $U_0 = 0$, where the detrimental factor is now the entropy of the random walk $\exp(-x^{*2}/t)$. Using self-consistently $w(t') = x^*(t')$, the compensation argument now leads to $x^* \propto t^{3/(4+d)}$, which is precisely the Flory result for the DP/KPZ problem. This Flory value can be obtained using a variational method, either with replicas [17] or without replicas [18]. In spirit, Eq. (6) is actually very close to the latter calculation. The value $b = 3/2$ can therefore be seen as a Flory value for this problem.

Returning now to the Sinai case, where the barrier height grows as $\sigma_F \sqrt{x^*}$, the self consistent compensation argument now leads to $\sigma_F \sqrt{x^*} \sim \sigma_\eta \sqrt{t/x^*}$, or $x^* \sim (\sigma_\eta/\sigma_F)\sqrt{t}$. The $\sqrt{t}$ behaviour is close to the numerical results shown in Fig. 2. However, as shown in Fig. 3, the dependence of x_{cm} on σ_F is found to be weaker than the $1/\sigma_F$ behaviour predicted by this simple argument, and closer to $1/\sigma_F^{3/4}$. We have also investigated numerically the case where the force derives from a fractional Brownian motion with a Hurst exponent H. The case $H = 1/2$ is the standard Sinai random walk potential considered above. An extension of the proliferation argument to this case predicts that $\nu^* = 1/(1 + 2H)$ for $H > 1/4$, reverting to the DP/KPZ value $\nu^* = 2/3$ for smaller values of H (i.e., when the potential is not 'confining' enough). As shown in Fig. 2, our numerical values for ν^* agree quite well with this prediction: for example $\nu^*(H = 3/4) \simeq 0.37$ and $\nu^*(H = 1/4) \simeq 0.65$.

In summary, we have investigated the competition between barrier slowing down and proliferation induced superdiffusion in a model of population dynamics in a random force field. The one-loop RG analysis close to the critical dimension $d_c = 2$ predicts a second order phase transition between a subdiffusive regime and a superdiffusive regime, at variance with our numerical results in $d = 1$ which suggest that both the Sinai and KPZ fixed points are unstable, while a new stable mixed fixed point appears. We have given a heuristic Flory like argument, which allows us to understand qualitatively the diffusive behaviour at this mixed fixed point, and also our results on proliferation assisted barrier crossing. This work can be extended in various directions: for example, a two-loop RG calculation would be interesting. One could also study the effect of non linear terms in the population equation, such as $-P^2$ or $\vec{\nabla} \cdot (P\vec{\nabla}P)$, and the role of a non zero external force $\langle F(x) \rangle$. It would be worth performing some numerical simulations of the barrier crossing problem and of the mixed model in $d = 2$.

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