A NESTED FACTOR MODEL FOR NON-LINEAR DEPENDENCES IN STOCK RETURNS

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Abstract. The aim of our work is to propose a natural framework to account for all the empirically known properties of the multivariate distribution of stock returns. We define and study a “nested factor model”, where the linear factors part is standard, but where the log-volatility of the linear factors and of the residuals are themselves endowed with a factor structure and residuals. We propose a calibration procedure to estimate these log-vol factors and the residuals. We find that whereas the number of relevant linear factors is relatively large (10 or more), only two or three log-vol factors emerge in our analysis of the data. In fact, a minimal model where only one log-vol factor is considered is already very satisfactory, as it accurately reproduces the properties of bivariate copulas, in particular the dependence of the medial-point on the linear correlation coefficient, as reported in Chicheportiche and Bouchaud (2012). We have tested the ability of the model to predict Out-of-Sample the risk of non-linear portfolios, and found that it performs significantly better than other schemes.

1. Introduction

Dependences among financial assets or asset classes stand at the heart of modern portfolio selection theories. Whatever the (concave) utility of an investor and its risk measure, diversification is profitable but optimal diversification is only reached if the underlying dependence structure is well understood.

For example, the well-known Markowitz theory (Markowitz, 1952, 1959; Bouchaud and Potters, 2003) of optimal portfolio design aims at finding the optimal weights \( w_i \) to attribute to each stock of a pool. It assumes that stock returns are correlated random variables \( x_i \), and that the optimizing agent has a “mean-variance” quadratic utility function in the form \( U(w) = E[x \cdot w] - \mu V[x \cdot w] \), with a parameter \( \mu \) controlling for risk-aversion level. It hence relies on the linear covariance matrix \( \rho = E[xx^\dagger] \) of the stock returns, and more importantly on its inverse \( \rho^{-1} \). Indeed, with no further constraints (budget, transaction costs, operational risk constraint, prohibition of short selling, etc.), the optimal weights are given by

\[
 w^*_\rho \propto \rho^{-1} g \equiv \Lambda^{-1} V^\dagger g
\]

where \( g \) is the vector of gain targets for the assets in the basket, and \( \rho = V\Lambda V^\dagger \) is the spectral decomposition of the covariance matrix with \( V \) being the square matrix of eigenvectors and \( \Lambda \) the diagonal matrix of eigenvalues.

Empirical estimates of \( \rho \) and its spectrum \( \Lambda \) are typically very noisy, and cleaning schemes need to be applied before inversion if one wants to avoid artificially enhancing the weights of low-risk in-sample modes — as made clear by the above expression of \( w^*_\rho \) in terms of \( \Lambda \) — that turn into high-risk realized out-of-sample modes.

All this is by now fairly standard practice, and several cleaning schemes have been designed, in view of modeling either the signal (parametric models, factor models, Principal Components Analysis), or the noise (RMT-based Laloux et al. (1999, 2000); Ledoit and Wolf (2004); Potters et al. (2005); El Karoui (2010); Bartz et al. (2012)), see also Tumminello et al. (2007); Potters and Bouchaud (2009).

However, it is now established that markets operate beyond the Gaussian, linear regime. For one thing, individual stock returns are well known to be non-Gaussian, and moments beyond the mean and variance have gained considerable interest (e.g. the excess kurtosis, or low-moment estimates thereof). But more importantly, stock returns are jointly not Gaussian: the structure of dependence between pairs of stocks is not compatible with the Gaussian copula, and as a consequence the penalty in the utility function should be more subtle than just the portfolio variance and include non-linear measures of risk (like tail events, quadratic correlations, etc.) in order to better fit the agent’s risk aversion profile. Only in a multivariate Gaussian setting can these non-linear dependences be fully expressed in terms of the linear correlations.

Non-linear dependences are also very important in the pricing and risk management of structured products and portfolios of derivatives. For example, the payoff of a hedged option has a V-shape with linear asymptotes and quadratic core, see Fig. 1. A portfolio of several such hedged options has thus a variance characterized by the absolute and quadratic correlations of the underlying stocks (gamma risk).
The correlations of these amplitudes, needed for estimating the risk at the portfolio level, are even noisier than linear correlations, whence the need for a reliable model of both linear and non-linear dependences.

In a previous article (Chicheportiche and Bouchaud, 2012), we showed that the joint distribution of daily returns of stocks is not "elliptical" either, which is to say that stocks are not exposed to a unique volatility volatility affecting all of them. We in fact ruled out all models with a single stochastic volatility $\sigma_i$, of the form

$$x_i = \sigma_i \epsilon_i,$$

with jointly Gaussian (and correlated) residuals $\epsilon_i$’s. This, we argued, revealed a finer structure in the non-linear dependences, and opened the way for a description taking into account several modes of volatility. However, our results also showed that any description in the form of individual volatilities

$$x_i = \sigma_i \epsilon_i,$$

with arbitrary dependences between the $\sigma_i$’s, would not be able either to explain successfully the empirical joint distribution. We focused in particular on the medial point of bivariate copulas, $C(\frac{1}{2}, \frac{1}{2})$ which is the probability that both variables are below their median value simultaneously.

All pseudo-elliptical models (defined by Eq. (3)) lead to a simple relation between the medial copula to the coefficient of linear correlation, see the discussion in Chicheportiche and Bouchaud (2012):

$$C_{ij}(\frac{1}{2}, \frac{1}{2}) = \frac{1}{4} + \frac{1}{2\pi} \arcsin \rho_{ij}$$

Said differently, the effective correlation

$$\rho_{ij}^{(B)} \equiv \cos(2\pi C_{ij}(\frac{1}{2}, \frac{1}{2}))$$

is equal to $\rho_{ij}$ for these models, whereas empirical data shows marked departures from this prediction, see Fig. 2. As discussed in Chicheportiche and Bouchaud (2012), the scatter plot of $\ln |\rho/\rho^{(B)}|$ vs $\rho$ for every stock pair is not concentrated around the elliptical prediction (dashed horizontal line crossing the y-axis at 0), but rather the average curve (black line) departs significantly from the prediction. Furthermore,

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1 The copula $C(u_i, u_j)$ of a random pair $(X_i, X_j)$ is the joint probability that the variables are below their marginal $(u_i, u_j)$-th quantiles respectively:

$$C(u_i, u_j) = P(x_i < \mathcal{P}^{-1}_{\alpha_i}(u_i), x_j < \mathcal{P}^{-1}_{\alpha_j}(u_j)),$$

where $\mathcal{P}_{\alpha}$ denotes the cumulative distribution function (CDF) of $x_i$.

2 The superscript (B) stands for “Blomqvist”, as $\rho^{(B)}$ is related to Blomqvist’s beta coefficient, see Blomqvist 1950.
the less correlated the pairs are, the farther they depart from an elliptical bivariate distribution, calling for a richer description than just amplitudes exposed to a common mode of fluctuations.

This, together with a comparison of empirical and theoretical values of other observables (e.g. copula diagonals), motivates a description of stock returns with several modes of volatility, but which excludes models where the \( \sigma_i \)'s can be decomposed multiplicatively into a market contribution \( \omega_i \), a sectorial contribution \( \tilde{\omega}_s \) (where stock \( i \) belongs to sector \( s \)), and a residual contribution \( \tilde{\omega}_i \) as:

\[
\sigma_i = \sigma_0 e^{\omega_i + \tilde{\omega}_s + \tilde{\omega}_i}.
\]

Instead, we proposed that additive non-Gaussian factors should be able to generate anomalous medial copula values, because of the interplay of factor kurtosis and residual kurtosis, as motivated by the toy model for \( C(\frac{1}{2}, \frac{1}{2}) \) presented in Chicheportiche and Bouchaud (2012).

The search for a theoretical description of multivariate dependences has led to the explosion of the literature on copulas. Several families of copulas have been proposed, beyond the elliptical one discussed above: Archimedean (Clayton, Frank, and others), Vine, Liouville, etc. Unfortunately, as emphasized in Chicheportiche and Bouchaud (2012), these copulas are often theoretical figments with no financial interpretation, and for that reason alone should be considered with suspicion. (These alternative copulas also fail to reproduce the empirical dependences of stock returns). We advocated in Chicheportiche and Bouchaud (2012) the need for constructing meaningful copulas, based on intuition and plausibility. The aim of the present paper is therefore to construct a general factor model, flexible enough to reproduce all the known stylized facts of the empirical joint distribution of stock returns, but still simple enough to be easily calibrated on data. We want our multivariate model of stock returns to be able to:

- Reproduce the structure of linear correlations with a small number of factors.
- Generate fat-tailed return series, with non-Gaussian factors and residuals;
- Allow for a dependence between the volatilities of the residuals and the volatilities of the factors, as observed in Cizeau et al. (2001), Allez and Bouchaud (2011).
- Reproduce the anomalous copula structure determined in Chicheportiche and Bouchaud (2012), in particular the diagonal and anti-diagonal and the medial point mentioned above, see Fig. 2. It was also noted there that highly correlated stock pairs are “more elliptical”, and that in periods of high turmoil like the financial crisis, stock pairs are both more correlated and more elliptical, revealing a strong exposure to a common mode of volatility.
- Predict the structure of non-linear (absolute values and quadratic) correlations with a reduced number of parameters, in order to clean the empirically measured dependence coefficients and allow for efficient out-of-sample risk control.

**Figure 2.** \( \ln |\rho/\rho_B| \) vs \( \rho \) for each stock pair, see Eq. [4]. This figure quantifies the departure of the medial point of the copula \( C(\frac{1}{2}, \frac{1}{2}) \) from the pseudo-elliptical benchmark defined by Eq. [3] for which the prediction is a straight horizontal line at 0 (dashed). Empirical values are shown as a scatter plot of all stock pairs (grey cloud) as well as bin averages (black line), for the 3 periods considered here. The “dominant volatility mode” prediction for the average behavior (see Sect. 4) is shown in red and agrees remarkably well with data, and captures correctly its time evolution. Note that \( \ln |\rho/\rho_B| \) tends to 0 when \( \rho \) increases, which shows that highly correlated stocks are more “elliptical”, i.e. indeed exposed to the same volatility mode.
Table 1. Economic sectors according to Bloomberg classification, with corresponding number of individuals for each period.

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As we shall show below, we achieve this with a “nested” factor model, i.e. standard factor model with volatilities of the factors that have themselves a multiplicative factor structure. We establish that the factor model for factor-volatilities requires one (or perhaps two) “dominant mode” (that also contributes to the volatility of the residuals) plus idiosyncratic contributions. Perhaps surprisingly, this dominant volatility mode is \textit{not} the volatility of the dominant (market) mode of the linear factor model.

Several very recent studies have reached conclusions that partly overlap ours. \cite{Kelly et al. 2012} document strong comovements of individual stock return volatilities. They find that the residuals of factor models (like \cite{Fama and French 1993} or Principal Components Analysis) exhibit a strong volatility dependence, which they capture using a one-factor (vol) model. This is to our knowledge the only attempt in the equity literature at describing volatility dependences in stock returns as a second-order effect, after removing linear correlations. However, it only focuses on \textit{residuals} volatility, and thus misses the \textit{factors} volatility correlations, as we reveal below. As mentioned when discussing Fig. 1, the options community is also much concerned by volatility dependencies for the risk description of options portfolios, and some studies have begun to address this issue. Notably, \cite{Engle and Figlewski 2012} acknowledge the comovements of the implied volatilities of options on individual stocks, and attempt to model their dynamics through an exposure to the VIX index. Very recently, \cite{Christoffersen et al. 2013} have proposed to calibrate a one-factor model with option data, using its predictions in terms of option pricing. They model stock prices dynamics with an exposure to a common stochastic market factor and stochastic idiosyncratic volatilities with correlated innovations. Our scope is rather multivariate analysis and bottom-up copula description. Our model focuses solely on cross-sectional properties and has for now no dynamical content, although this is a natural next step, which is easy to do (at least conceptually). Our nested factor is more complete than the ones mentioned above, in that it is able to reproduce more stylized facts, in particular the common structure of factors volatilities and idiosyncratic volatilities, and the subtle properties of the bivariate empirical copulas.

\textbf{Data set.} We will construct and calibrate our model on the daily close-to-close log-returns of stock prices of companies that are present in the S&P500 index during the whole of the period studied. We will be considering three periods of roughly 5 years for the empirical study and the model calibration: before the financial crisis (Jan, 2000 – Dec, 2004); during the financial crisis (Jan, 2005 – Dec, 2009); after the financial crisis (Aug, 2009 – Dec, 2012). A longer dataset is used for the sliding windows procedure of In-sample/Out-of-sample testing in the last section: there we consider the ten years period 2000–2009.

It will be useful to group the companies according to their sector of activity, in order to see if decipherable patterns appear. We will make use for that purpose of Bloomberg’s classification as summarized in Tab. 1.

\textbf{Outline.} This paper is made of an introduction, four sections and a conclusion. In Section 2 we study the linear correlations of pairs of stocks and discuss the design and estimation of a factor model for their description. The non-linear dependences of factors and residuals generated by the calibration of the model are studied in Section 3 and motivate the specification of the volatility content of the model that we present in Section 4. The resulting nested factor, non-Gaussian model is calibrated, and Section 5 is dedicated to an Out-of-Sample stability analysis, that validates the usefulness of our description of non-linear dependences. Methodological points for the estimation of the model parameters and the assessment of the model’s performance are provided in appendices.
2. Linear factors

We first recall the definition and basic properties of a simple one-level factor model for the joint description of the stock returns \( x_i \) of \( N \) firms, as a combination of \( M \) shared factors \( f_k \):

\[
x_i = \sum_{k=1}^{M} \beta_{ki} f_k + e_i.
\]

(5)

The weight \( \beta_{ki} \) parameterizes the linear exposure of stock \( i \) to factor \( k \). At this stage, we do not yet specify the statistical properties of the factors \( f_k \) and residuals \( e_i \), except that we impose that they are linearly uncorrelated, i.e.:

\[
\begin{align*}
E[f_k f_\ell] &= \delta_{k\ell} \quad (6a) \\
E[e_i e_j] &= \delta_{ij} \left(1 - \sum_{\ell} \beta_{i\ell}^2\right) \quad (6b) \\
E[f_k e_j] &= 0. \quad (6c)
\end{align*}
\]

Written in matrix form, the factor model reads:

\[
X = F \beta + E,
\]

(5')

with unit-variance factors \( F \) \((T \times M)\), exposures \( \beta \) \((M \times N)\) of every stock to every factor, and orthogonal residuals \( E \) \((T \times N)\). In this way, the residuals \( e_i \) can be understood as idiosyncratic shocks and all the linear dependence is accounted for by the factors. The predictions of the model in terms of covariances of the returns \( x_i \) do not need additional assumptions, and only depend on the matrix of linear weights \( \beta \). Assuming that the returns are normalized to have unit variance, one has:

\[
\rho_{ij} = E[x_i x_j] = \begin{cases} 
(\beta^\dagger \beta)_{ij}, & i \neq j \\
1, & i = j
\end{cases} \quad (7)
\]

The above linear factor model is of course the workhorse of the econometric literature. However, there are two subtle points about it that need to be clarified.

1. In the econometric literature, one often assumes that the set of explanatory factors is known. The time series of these factors \( F_{tk} \) are then inputs of the estimation problem, whereas the elasticities \( \beta \) are the output of the linear regression. Here, we will rather determine the weights \( \beta \) in such a way that the empirical correlation matrix is as close as possible to the one predicted by an \( M \)-factor model, Eq. (7):

\[
\arg\min \left\| \frac{1}{T} X^\dagger X - \beta_F^\dagger \beta_F \right\|_{\text{off-diag}}. \quad (8)
\]

When the weights \( \beta_F \) are known, it is possible to design a different identification scheme that generates orthogonal residuals. Consider indeed the date-by-date regression of the \( N \) returns on the \( M \) (freshly estimated) weights \( \beta_F \):

\[
X_t = F_t \beta_F + E_t. \quad (9)
\]

The regression parameters to be estimated are then the value of the \( M \) factors \( F_t \), for date \( t \). A GLS solution of the regression then yields the wanted factors and residual series. It is only approximate in the sense that \( \frac{1}{T} E^\dagger E \) is only “as close as can be” to a diagonal matrix, and \( \frac{1}{T} F^\dagger F \) is only approximately \( I_M \).

2. The linear factor methodology looks superficially similar to a standard Principal Components Analysis (PCA). We expand in Appendix \( \text{A} \) on the similarities and differences between the two points of view. In fact, we use the results of the PCA as a starting point for the numerical optimization program defined by Eq. (8).

We will show in Sect. \( \text{E} \) that our factor model approach in fact outperforms the PCA approach by more than 5% when it comes to comparing the out-of-sample risk of optimal portfolios constructed using these two methods with an in-sample risk almost unchanged, see Fig. \( \text{H} \) below.

We have calibrated the linear weights \( \beta_F \) on the three data sets by solving Eq. (8), using \( M = 10 \). We will discuss in details in the next section the properties of the factors time series \( F_t \), and residual time series \( E_t \); we will show in particular that while these time series are indeed approximately uncorrelated,
strong non-linear dependencies remain, and this will suggest the building blocks of our nested factor model.

3. Properties of the reconstructed factors and residuals

The calibration procedure of the linear model (5) worked out in the previous section outputs the series of factors $F_t$ and residuals $E_t$. The average linear correlation over all pairs of factors is indeed very small, $\approx 1.2 \cdot 10^{-4}$, with a standard-deviation $55 \cdot 10^{-4}$, similar for all periods. The average linear correlation over all pairs of residuals is around $(2.5 \pm 45) 10^{-3}$, and the average cross-correlation between factors and residuals is of the order $10^{-4}$. Although not exactly zero these small numbers are clearly within the noise (which is larger than $1/\sqrt{T} \approx 3 \cdot 10^{-2}$ because of volatility persistence) and illustrate that the resulting series of factor returns and residuals are to a very good approximation all uncorrelated.

This does not mean however that they are independent. Indeed, we will show in this section that all the volatilities of these series are strongly dependent. We will therefore enhance the factor model defined in Eqs. (5-6) by a characterization of the non-linear dependencies among the $f$’s and the $e$’s. Note that the factors and residuals are expected (and found) to be strongly non-Gaussian. In fact, we will model the volatilities as approximately log-normal processes (but see below).

The non-linear properties of the reconstructed factors and residuals can be investigated through the correlations of absolute values, or squares, etc. Since this choice is to some extent arbitrary, we have defined the generalized non-linear correlations for factors and residuals as:

$$C_{k\ell}^F(p) = \frac{1}{p^2} \ln \frac{\langle |F_{k\ell}|^p \rangle}{\langle |F_{k\ell}| \rangle^p} \quad \text{(10a)}$$

$$C_{ij}^F(p) = \frac{1}{p^2} \ln \frac{\langle |E_i E_j|^p \rangle}{\langle |E_i| \rangle^p} \langle |E_j| \rangle^p \quad \text{(10b)}$$

$$C_{k\ell}^E(p) = \frac{1}{p^2} \ln \frac{\langle |F_{k\ell} E_{kj}|^p \rangle}{\langle |F_{k\ell}| \rangle^p \langle |E_j| \rangle^p} \quad \text{(10c)}$$

for any value of $p > 0$. The normalization with $p^{-2}$ ensures that these coefficients tend to a finite value when $p \to 0$, and they would actually be independent of $p$ for multivariate log-normal volatilities. Large values of $p$ lead to very noisy estimators, so we restrict below to $p \in (0, 2]$.

As an example, we show in Fig. 3 the off-diagonal matrix elements of the factor-factor correlations (10a) for $M = 10$. Each figure corresponds to a value of $k$, the different curves represent the values of Eq. (10a) for different $\ell \neq k$, as a function of $p$. We observe that a) these correlations clearly are non-zero, whereas a factor model with Gaussian statistics would give zero (since in this case, factors would not only be uncorrelated but independent); and b) The concavity of the curves is a signature of non-Gaussianity in log-volatilities, while their splitting (in particular as $p \to 0$) reveals a complex structure that we will uncover using a model in Section 4 below.
Because of the large number of residuals, the above “naked eye” analysis is not possible for the factors-residuals and residuals-residuals correlations, for which it turns much more convenient to use a spectral approach in terms of singular value decompositions (which boils down to eigenvalues/eigenvectors for symmetric objects like $C^{rr}$ and $C^{ff}$). In terms of singular values/eigenvalues, we find that for all $p \in (0, 2]$, two of them clearly stand out, while the rest stands within a noisy “bulk”. The largest one is furthermore a factor 3 to 5 larger than the second one, suggesting that a one- (or two-) factor model for the log-volatilities should provide a good description of the data (see below).

The corresponding eigenvectors are to a good approximation independent of $p$. The components of two dominant eigenvectors of $C^{ff}$ are shown in Fig. 3 (left), for the period 2000–2009 and averaged over $p$. As expected, the largest eigenvalue has an associated vector approximately uniform over the $M = 10$ factors. Zooming into sub-periods, this mode seems to be impacting/impacted by the financial sector more strongly in the 2005–2009 period (containing the financial crisis) whereas in other periods it is almost uniformly spread over sectors. The second eigenvalue has a non-trivial structure which is less robust in details, although the overall pattern is similar for the different sub-periods.

The eigenvectors of $C^{rr}$ are of dimension $N$ (the number of stocks) and thus less easy to visualize. We show in Fig. 3 (right) the components of its two dominant eigenvectors in a representation where stocks are grouped according to their Bloomberg classification (see the grey vertical lines separating these sectors). Note that the finance sector plays a special role here: its weight is larger in the first eigenvector, while the second eigenvector is to a first approximation “finance against all”. Zooming again into sub-periods, the financial sector is clearly a stand-alone mode of fluctuations in the crisis period. In the pre-crisis period, the second relevant mode is rather composed of commodities. Indeed, the second eigenvector features the opposition of utilities, energy and communications against the rest. In the post-crisis period, on the other hand, there is no clear signature of the structure of the second eigenvector of $C^{rr}$.

Finally, the singular value decomposition of the mixed correlation matrix $C^{ff}$ is consistent with the above findings: the two dominant left-eigenvectors are nearly identical to the two dominant eigenvectors of $C^{rr}$ while the two dominant right-eigenvectors are nearly identical to the two dominant eigenvectors of $C^{rr}$. This confirms that we only need to focus on these four eigenvectors, two of dimension $M = 10$, two of dimension $N$.

4. A factor model for volatilities

The spectral analysis of the previous section suggests the existence of two volatility factors that drive the amplitude of both the factors $f_k$ and the residuals $e_i$.\footnote{A third eigenvalue of $C^{rr}$ might in fact be significant, but we will discard it altogether in the present study.}
More concretely, we propose the following multiplicative model for the volatilities, which defines our nested factor model:

\[ f_k = \epsilon_k \exp(A_{k0} \Omega_0 + A_{k1} \Omega_1 + \omega_k) \]  
\[ \epsilon_j = \eta_j \exp(B_{j0} \Omega_0 + B_{j1} \Omega_1 + \tilde{\omega}_j), \]

where the \( \Omega \)'s are stochastic factor log-volatilities and the \( \omega \)'s are stochastic “idiosyncratic” log-volatilities (all independent of each other and independent of the Gaussian noises \( \epsilon \)'s and \( \eta \)'s). The parameters \( A \)'s and \( B \)'s weight the contribution of every volatility mode. In particular, we expect \( A_{k0} \) and \( B_{j0} \) to be given by the dominant eigenvectors of \( C_A \) and \( C_B \), respectively, and \( A_{k1} \) and \( B_{j1} \) by the second eigenvectors.

In the next subsection, we first estimate a minimal model with a single volatility driver, \( \Omega_0 \).

4.1. A dominant volatility mode.

4.1.1. Definition. The minimal improvement over the independent factors assumption, while keeping uncorrelated factors, is to allow for a single common source for the fluctuation of amplitudes, i.e. set \( A_{k1} = B_{j1} = 0 \), \( \forall k, j \) in the above equations:

\[ f_k = \epsilon_k \exp(A_{k0} \Omega_0 + \omega_k) \]  
\[ \epsilon_j = \eta_j \exp(B_{j0} \Omega_0 + \tilde{\omega}_j), \]

with \( \epsilon_k, \eta_j \) Gaussian, with variance such that Eqs. \( \ref{eq:gaussian} \) hold.

Because this model is already a level of complexity higher than the standard linear factor model, it is worthwhile to insist on the intuitive meaning of the different log-volatility factors:

\( \Omega_0 \) = dominant and common driver of log-volatility across all factors and residuals,

\( \omega_1 \) = idiosyncratic log-volatility of the market mode \( f_1 \) (a.k.a. the index), net of \( \Omega_0 \),

and the subsequent \( \omega_k, \tilde{\omega}_j \) \( (k > 1) \) characterize the “residual volatilities” not explained by the common driver \( \Omega_0 \) in the amplitude of the factors \( f_k \) and residuals \( \epsilon_j \). Note in particular that the dominant log-volatility factor \( \Omega_0 \) cannot be identified with the log-volatility of the dominant market mode \( f_1 \) in the linear factor model!

The model is completely characterized from a probabilistic point of view when the law of the log-volatilities is specified. The \( p \)-dependence of the curves in Fig. 3 suggests that the non-Gaussianity in the log-volatilities is approximately homogeneous across the factors, and thus possibly due to the common volatility driver \( \Omega_0 \) alone, while the residual volatilities \( \omega_k \) and \( \tilde{\omega}_j \) can be taken as Gaussian (at least in a first approximation). For \( \Omega_0 \), we set:

\[ E[\Omega_0] = 0 \]  
\[ E[\Omega_0^2] = 1 \]  
\[ E[\Omega_0^3] = \zeta_0 \]  
\[ E[\Omega_0^4] = 3 + \kappa_0. \]

At this stage, a recap is probably useful. Our nested factor model with a single volatility mode (defined by Eqs. \( \ref{eq:nested} \)) contains the following parameters:

- \( MN \) linear weights \( \beta_{ki} \) (already estimated, see Sect. 2);
- \( M \) coefficients \( A_{k0} \) and \( N \) coefficients \( B_{j0} \) giving the exposure of factors and residuals to the common volatility mode \( \Omega_0 \) (of unit variance);
- The standard-deviations \( s_k \) and \( \tilde{s}_j \) of the residual Gaussian log-volatilities \( \omega_k \) and \( \tilde{\omega}_j \);
- And finally the skewness \( \zeta_0 \) and kurtosis \( \kappa_0 \) of the dominant volatility mode \( \Omega_0 \).

So there are overall \( MN + 2(N + M) + 2 \) parameters, for a dataset of size \( NT \). More importantly the number of parameters is only marginally increased with respect to a typical linear factor model (where only the \( MN \) linear weights enter into account): only \( 2(N + M + 1) \) new parameters, intended to improve the description of all \( N(N - 1)/2 \) pairwise dependences coefficients.

The calibration procedure, that allows to determine these \( 2(N + M + 1) \) new parameters, is detailed in Appendix 3.

4.1.2. Results of the calibration. The calibration results are given graphically in Figs. 5, 6, 7, where we show, separately for each sub-period, the estimated parameters \( A_{k0} \) and \( B_{j0} \). For the reason discussed in Appendix 3 they turn out to be very close to the first eigenvector of the corresponding matrix of “log-abs” correlations discussed in Sect. 3 above. Of particular interest are the ratios \( B_{j0}/A_{k0} \), which are found to be on average less than unity (0.79 in 2000–2004, 0.49 in 2005–2009 and 0.40 in 2009–2012). This means that the dominant volatility mode affects both the index volatility and the residual volatilities, as noted in Cîrstea et al. (2001), but in a weaker way for the latter. This was already observed in Allez and Bouchaud (2011), see their Fig. 3.

We also show in Figs. 5, 6, 7 the parameters \( s_k \) and \( \tilde{s}_j \). Note that some factors \( k \) seem to have their volatility entirely explained by the common driver \( \Omega_0 \) so that there is no residual volatility left.
The estimated values of the non-Gaussianity parameters of the log-volatilities are reported in Tab. 2. Noticeably, the kurtosis of the common driver $\Omega_0$ is found to be negative in every period: the log-volatility is less kurtic than a Gaussian, which is a rare finding in financial time series analysis! This was already revealed by the concavity of the curves in Fig. 3.

4.1.3. Dynamics of the common volatility mode. Interestingly, we are now in position to reconstruct the time series of the common volatility mode $\Omega_{t0}$ out of the model equations and the estimated parameters.
Similarly to what was done in Sect. E to recover the series of linear factors, we perform here two date-
by-date regressions motivated by the Eqs. (12):
\[
\ln |F_{tk}| - \langle \ln |F_{tk}| \rangle = \Omega_{t0} A_{k0} + s_k \omega_{tk} \\
\ln |E_{tj}| - \langle \ln |E_{tj}| \rangle = \Omega_{t0} B_{j0} + \bar{s}_j \omega_{tj}
\]
Whereas the first regression is performed over only the \( M \) variables \( A_{k0} \), the second one is realized over
the \( N \) variables \( B_{j0} \) and thus leads to much less noisy estimates of \( \Omega_{t0} \) (we will always use the second
determination in the following). The overlap of the time series of \( \Omega_{t0} \) estimated with the two regressions is
nevertheless quite good, with a correlation coefficient between 0.55 and 0.75 depending on the period
studied. We show in Fig. 8 the time series \( \exp (A_{t0} \Omega_{t0}) \) reconstructed from the procedure above after
estimation of the parameters, that we compare to the absolute value of the market factor \( F_{t1} \).

An obvious next step would be to calibrate a dynamical model (GARCH or stochastic volatility) to
account for the temporal evolution of \( \Omega_{t0} \).

As consistency checks of both the quality of the model and the estimation procedure, we now analyze
the model prediction with the estimated parameters and compare them with empirical measurements
of the same quantities. Of particular interest are the quadratic correlations and the diagonal copulas,
whose anomalies observed in a previous study \cite{chicheportiche2012} actually motivated the
present model.

4.1.4. Quadratic correlations. The quadratic correlations can be explicitly computed from the model
definition, and write:
\[
\mathbb{E}[x_t^2 x_j^2] = \sum_{kk} \left( \beta^2_{k0} \beta^2_{j0} + 2 \beta_{k0} \beta_{j0} \beta_{k0} \beta_{j0} \right) \Phi_0(A_{k0}, A_{j0}; 2) \left( \frac{1}{2} \cdot 3 \cdot \Phi_G(s_k, s_j; 2) \right)^{\delta_{kk}} \\
+ (1 + 2 \delta_{ij}) \left( 1 - \sum_{\ell} \beta^2_{\ell0} \right) \sum_{k} \beta^2_{k0} \Phi_0(A_{k0}, B_{j0}; 2) \\
+ (1 + 2 \delta_{ij}) \left( 1 - \sum_{\ell} \beta^2_{\ell0} \right) \sum_{k} \beta^2_{k0} \Phi_0(A_{k0}, B_{j0}; 2) \\
+ \left( 1 - \sum_{\ell} \beta^2_{\ell0} \right) \left( 1 - \sum_{\ell} \beta^2_{\ell0} \right) \Phi_0(B_{j0}, B_{j0}; 2) \left( 3 \Phi_G(s_i, s_j; 2) \right)^{\delta_{ij}},
\]
where \( \Phi_0 \) is defined in Appendix B. When all parameters \( A, B, s \) are zero, the prediction for Gaussian
factors and residuals is retrieved: \( \mathbb{E}[x_t^2 x_j^2] = 1 + 2 \mathbb{E}[x_t x_j]^2 \). We illustrate in the left panel of Fig. 9 a
scatter plot of the left-hand side (calibrated) versus the right-hand side (empirical) of Eq. (13), for all
periods. They show a good agreement of model and sample quadratic correlations. Furthermore, the
middle and right panels of the same figure illustrate the fact that the pairs of stock returns cannot be
described by a bivariate Student distribution, for which a regular curve should be observed instead of the
scattered cloud in the plane of quadratic vs linear correlations. This conclusion was already reached in Chicheportiche and Bouchaud (2012), and is made precise by the present nested factor model.\(^5\)

4.1.5. Copulas: medial point and diagonals. The middle point \(C(\frac{1}{2}, \frac{1}{2})\) of the copula, which was shown on Fig. 2 to be incompatible with any elliptical prediction, is also very well captured by our model, with no further ingredients. Although an analytical expression relating \(C(\frac{1}{2}, \frac{1}{2})\) to the model parameters is out of reach, it is possible to reproduce its predicted value by simulating long time series according to the model with estimated parameters.\(^6\) The results are in remarkable agreement with the data (see Fig. 2), and emphasize the capacity of our non-Gaussian factor model to cope with the non-trivial behavior of the medial point of the copula.

This is confirmed and in fact strengthened by the analysis of the bivariate copulas along the whole diagonals. Fig. 10 compares empirically measured and model-predicted values of the quantities

\[
\Delta, p = \frac{C(p, p) - CG(p, p)}{p (1 - p)} \quad \text{and} \quad \Delta, p = \frac{C(p, 1 - p) - CG(p, 1 - p)}{p (1 - p)} \quad (14)
\]

versus \(p\), for several values of the linear correlation over 2000–2004 (similar plots for other periods are produced in Chicheportiche (2013)). A direct visual comparison reveals that the main non-trivial qualitative features of the empirical diagonal copulas are well reproduced by our model. For example, the evolution of the concavity as \(p\) changes, the behavior in the tails, and the medial-point behavior as discussed above. Plots of similar quality have been obtained for other sub-periods as well.

One may note however that the asymmetry \(u \leftrightarrow 1 - u\), visible in the graphs, is not reproduced by our fully symmetric model, and would require accounting for the leverage effect, i.e. cross-correlations between the linear factors and residuals \(f_k, e_j\), and the volatility factors and residuals \(\Omega_0, \omega_k, \hat{\omega}_j\).

Before proceeding to the out-of-sample evaluation of the model in Sect. 5, we briefly present how the model can be improved (though marginally) by introducing a second volatility driver \(\Omega_1\), as suggested by the spectral analysis of Sect. 3.

4.2. A second volatility driver. The spectral analysis of the factor and residual absolute correlations has revealed that there exists a small but significant second mode of volatility. The model in Eq. (12) can be improved accordingly in order to account for this additional source of collective amplitude fluctuations, see Eq. (11) above.

The whole estimation procedure runs identically. However, for the determination of the parameters \(A_{k0}\) and \(A_{k1}\), the reduced number of observations \((M(M - 1)/2\) factor-factor correlations, times \(8\) values of \(p\)) provides only a low resolution, and the minimization program does not succeed in distinguishing the two volatility drivers: it outputs an hybrid where both \(\Omega_0\) and \(\Omega_1\) contribute to the same mode. In order to break the degeneracy and “orthogonalize” the modes, we add an overlap term \((\sum_k A_{k0} A_{k1})^2\) in the cost function Eq. (27).

As an example, we report in Fig. 11 the results for the period 2000–2004. As expected, the parameters \(A_{k0}\) and \(A_{k1}\) are very close to the first two eigenvectors of the factor-factor “log-abs” correlation matrix, and the parameters \(B_{j0}\), \(B_{j1}\) look like the first two eigenvectors of the residual-residual matrix. Clearly, taking this additional second volatility driver into consideration improves the theoretical description of the returns. We illustrate this on Fig. 11(c) where we show how \(\Omega_0\) and \(\Omega_1\) contribute to the volatility of the market mode of linear correlation, \(f_1\).

5. Out-of-sample analysis

All the results presented above are “in-sample”, in the sense that we have shown the predicted dependence coefficients with estimated parameters on a period and compared them to the realized coefficients in that same period. The ultimate test for a model that aims at describing joint financial returns (and more generally of any risk model), is to improve “out-of-sample” predictions, i.e. use a model calibrated on a period to predict some quantity in a subsequent period.

\(^5\) Notice that the choice of \(p\) in the estimation procedure of the parameters \(B_{j0}\) and \(\tilde{s}_i\) is important here. Estimation biases and errors are in practice different for low moments \(p \approx 0.2\) or high moments \(p \approx 2\). Obviously, best fits for the quadratic correlations are obtained with \(p = 2\) since in this case the same quantities appear in Eq. (13) and in the loss function \(28\).

\(^6\) The non-Gaussian series of log-volatility \(\Omega_0\) is generated as independent realizations of a Beta distribution whose coefficients are determined so that the first four moments match those of \(\Omega_0\). This class of distributions allows for negative kurtosis. It is known that the realizations of volatility exhibit strong persistence, a characteristic that our simulated series do not reproduce. This however does not generate a bias in the obtained coefficients, but rather makes them “not noisy enough”.
We will test the different models through the predicted correlation matrix. For the (linear) correlation matrix of the returns themselves, this has been the subject of many papers in the literature already, see [Laloux et al., 1999; 2000; Ledoit and Wolf, 2004; Potters et al., 2005; Tumminello et al., 2007; El Karoui, 2005; Bartz et al., 2012; Potters and Bouchaud, 2009]. Even if this is not the primary aim of the present study, we will first test the ability of our linear factor model to correctly predict the out-of-sample risk of optimal linear portfolios. We will then turn to the case of a portfolio of non-linear assets (absolute values), which has not been considered so far in this context (to the best of our knowledge). We will show that our “dominant volatility mode” framework outperforms other natural models for predicting out-of-sample risk.

We will consider a long period 2000–2009 on which we perform an In-sample/Out-of-sample analysis over sliding windows (N = 262 returns series are kept, see Tab. 1). We rely on the procedure introduced by Potters and Bouchaud (2009), that we reproduce for convenience in Appendix C.

5.1. Linear correlations. We first revisit the standard Markowitz problem, attempting to minimize the out-of-sample risk of an optimal portfolio constructed using as input different correlation matrices:

- Empirical: the in-sample raw correlation matrix,
  \[ \rho^{(1)}_{Emp}(\tau) = \frac{1}{T^2} \sum_{t'=\tau-T^S}^{\tau-1} X_{t'} \cdot X_{t'}; \]
- Ledoit-Wolf [Ledoit and Wolf, 2004]: the convex combination
  \[ \rho^{(1)}_{LW}(\tau) = \alpha \rho^{(1)}_{Emp}(\tau) + (1 - \alpha) \overline{\rho}^{(1)}(\tau), \]
  where \[ \overline{\rho}^{(1)}_{ij} = \begin{cases} 1 & ; i = j \\ 1/N(N-1)/2 \sum_{j' < j, i' < i} [\rho^{(1)}_{Emp}]_{i'j'} & , i \neq j \end{cases}. \]
  This corresponds to a “shrinkage” of the noisy sample correlation matrix toward its rank-one approximation.
- Clipped: \[ \rho^{(1)}_{Clipped}(\tau) \]
  retaining only the M eigenmodes of \( \rho^{(1)}_{Emp}(\tau) \) with largest eigenvalues, and adjusting all remaining eigenvalues to \((N - \sum_{i=1}^{M} \lambda_i)/(N - M)\) in order to conserve the trace;
- MultiFactor: the improved solution of Eq. (22), \[ \rho^{(1)}_{MF}(\tau) = \beta_F(\tau)^1 \beta_F(\tau) \]
  (for off-diagonal elements), for several values of the number of factors \( M \).

All these scenarios can furthermore be compared to the benchmark of a full-rank pure noise Wishart correlation matrix. In this case, Random Matrix Theory predicts the values of the average in-sample and out-of-sample risks, in the limit of large matrices with quality factor \( q = N/T^S \) (Potters and Bouchaud, 2009):

\[ \langle R_{RMT}^2 \rangle_{IS} = R_{True}^2 (1 - q) \quad \text{and} \quad \langle R_{RMT}^2 \rangle_{OS} = R_{True}^2 / (1 - q). \]

Moreover, the true risk (i.e. the value of \( R^2 \) when the optimal weights are determined using the correlation matrix of the process that generates the realized returns \( X_i/\sigma_i^{(1)} \)) can be shown to be \( R_{True}^2 = 1 \) with the definition (18).

We show graphically the results of the testing procedure on Fig. 12. In-sample and Out-of-sample average risks of every cleaning scheme are plotted parametrically with a control parameter \( \alpha \) (equal to \( M/N \) for Clipping and MultiFact), where averages are performed over sliding windows (\( \tau \)).

When only a very reduced number of factors (\( M \approx 1, 2, 3 \)) is kept, eigenvalue clipping performs better (although quite bad), and similarly when keeping also the very last modes: this is because the linear factors are only good when the eigenmodes are statistically significant, on the left and right of the RMT noise bulk. In the limit \( \alpha \to 1 \) (i.e. \( M = N \)) all cleaning schemes collapse to the risk values associated with the raw “Sample” correlation matrix. The benchmark RMT prediction is shown for reference.

In the intermediate regime, the Out-of-sample risk is minimal because the marginal gain in signal is higher than the marginal risk increase due to added “false information”. In this case, it turns out that the “factor model” procedure worked out in Appendix A provides an improved determination of average Out-of-sample risk. In fact, the inset of Fig. 12 shows that the relative gain

\[ \langle (\gamma_{Clipped}^2) - (\gamma_{MF}^2) \rangle / \langle (\gamma_{Clipped}^2) - \gamma_{True}^2 \rangle \]

(15)
can reach up to 6–7%, while not dramatically increasing over-fitting: the In-sample risk is only slightly artificially lowered.

Therefore, although our aim was to set up a model that would describe faithfully the non-linear dependences between stocks, we find that the first step of our procedure, namely the calibration of a
factor model to capture the linear correlations, leads to the best cleaning procedure so far (at least for the out of sample risk criterion we use here).

5.2. Absolute correlations. We now turn to the core property of our model: its ability to capture non-linear dependences. We have already shown that the model is able to reproduce, after calibration, several empirically observed quantities like the copula, and want now to perform an out-of-sample assessment of the volatility-driven dependence in the absolute correlations. The definitions of the gain predictor and the risk measure $R$ are identical to Eqs. (30) and (31) respectively (see Appendix C), with now

$$Y_{ti} = \frac{|X_{ti}| - \langle|X_{ti}|\rangle_t}{\sqrt{\langle(|X_{ti}| - \langle|X_{ti}|\rangle_t)^2\rangle_t}}.$$ 

The different cleaning schemes considered are:

- **Empirical**: the in-sample raw correlation matrix,
  $$\rho^{(\text{Emp})}_{\text{Lin}}(\tau) = \frac{1}{T_{\text{IS}}} \sum_{t' = \tau - T_{\text{IS}}}^{\tau - 1} Y_{t'} \cdot Y_{t'};$$

- **Ledoit-Wolf** [Ledoit and Wolf, 2004]: the convex combination
  $$\rho^{(\text{LW})}_{\text{Lin}}(\tau) = \alpha \rho^{(\text{Emp})}_{\text{Lin}}(\tau) + (1 - \alpha) \rho^{(\text{Out})}_{\text{Lin}}(\tau),$$
  similarly to the linear case. This corresponds to a “shrinkage” of the noisy in-sample cor-abs matrix toward its rank-one approximation.

- **Clipped**: $\rho^{(\text{Clip})}_{\text{Lin}}(\tau)$ retaining only the $M$ eigenmodes of $\rho^{(\text{Emp})}_{\text{Lin}}(\tau)$ with largest eigenvalues, and adjusting all remaining eigenvalues to $(N - \sum_{i=1}^{M} \lambda_i)/(N - M)$ in order to conserve the trace; 

- **Gaussian factors**: the Gaussian prediction $\rho^{(\text{Gauss})}_{\text{Lin}}(\tau)$ obtained as the sample absolute correlations of long time series simulated according to the $M$-factor model where all volatility parameters $A$, $B$ and $s$ are set to 0.

- **Multifactor (model)**: the model prediction $\rho^{(\text{Mf})}_{\text{Lin}}(\tau)$ obtained as the sample absolute correlations of long time series simulated (an analytic expression of absolute correlations is out of reach) according to the $M$-factor model with one volatility mode.

Notice that the meaning of $M$ is not comparable in all cleaning schemes: while for the “clipped eigenvalues” it corresponds to the number of relevant modes in the matrix of absolute correlations, for the multi-factor models it instead counts the the number of linear factors. This can be seen immediately on Fig. [13] where the red curve corresponding to “Clipping” has the usual U-shape, while the blue and magenta points corresponding to “multi-factor” saturate as $M$ increases above $\approx 30$, a threshold above which letting additional linear factors barely affects the volatility dependences.

More importantly, this figure shows that multi-factor models offer a better optimal Out-of-sample risk together with less In-Sample over-fitting. The role of volatility dependences is revealed by the much better performance of the final non-Gaussian multi-factor level over the Gaussian multi-factor cleaning scheme. This is emphasized in the inset plot of Fig. [13] representing the over-performance ratio

$$\frac{\langle R^2_{\text{FGG}} \rangle - \langle R^2_{\text{FGG}} \rangle}{\langle R^2_{\text{FGG}} \rangle - \langle R^2_{\text{true}} \rangle}. \quad (16)$$

The non-Gaussian model performs always better than the Gaussian model.

5.3. How many factors should be kept? The number $M$ of linear factors in the description (5) is an important input of the model. The intuition that statistical factors are somewhat related to economic sectors does not stand the identification of algebraic modes of fluctuations to sectorial or other macro-economic factors, beyond the first two or three modes. Still, even if there is no one-to-one identification, the number of sectors can be regarded as a reasonable prior for $M$. In our calibration, we have retained $M = 10$ factors corresponding to the number of Bloomberg sectors plus one, with satisfactory results at reproducing the main empirical stylized facts.

A more convincing determination of $M$ is reached by reconsidering the above results on the In-Sample/Out-of-Sample risk test for linear portfolios. From a general standpoint, we know that there must exists an optimal number of parameters, for which the model fits reasonably the data and avoids over-fitting, i.e. is stable when applied Out-of-Sample. Adjusting the ratio $\alpha = M/N$ allowed us to find an optimal configuration where the Out-of-Sample risk is minimized while the In-Sample risk is not artificially lowered. A value of $\alpha \approx 0.1$ is found to be optimal for the standard (linear) Markowitz problem, while the the risk associated with absolute returns is lowered by our non-Gaussian factor model whatever
the number of linear modes kept. This suggests that the optimal number of factor to be kept is \( M \approx 24 \) for the 262 stocks considered here.

6. Conclusions

Finding a faithful mathematical representation of the multivariate distribution of stock returns in a given market is one of the unsolved problems in quantitative finance. Copula-based research efforts have investigated a host of different possibilities, with disappointing results — both because the proposed copulas are not able to grasp the clear “stylized facts” evidenced by empirical studies, and because most of these copulas lack financial motivation and intuition. Among these stylized facts, two are particularly striking (Cizeau et al. [2001] Allez and Bouchaud [2011] Chicheportiche and Bouchaud [2012]):

- The market factor (index) volatility is strongly correlated with the volatility of residuals, even when the index and residuals are by construction uncorrelated. This naively suggests a multiplicative structure for stock returns, schematically as \( r = \sigma (\beta F + e) \).
- However, empirical copulas are incompatible with the above multiplicative (pseudo-elliptical) structure. In particular, medial-points of bivariate copulas have a non-trivial dependence on the linear correlation, that rules out this family of dependence, except for very highly correlated pairs of stocks.

The aim of our work was to propose a natural framework to account for all the empirical properties of the multivariate distribution of stock returns. We defined a “nested factor model”, where the linear factors part is standard (apart from the calibration procedure), but where the log-volatility of the linear factors and of the residuals are themselves endowed with a factor structure and residuals. We proposed a calibration procedure to estimate these log-vol factors and the residuals. We found that whereas the number of relevant linear factors is relatively large (10 or more), only two or three log-vol factors emerge in our analysis of the data. In fact, a minimal model where only one log-vol factor is considered is already very satisfactory, as it accurately reproduces the properties of bivariate copulas, in particular the subtle medial-point properties mentioned above. We have tested the ability of our model to predict Out-of-Sample the risk of non-linear portfolios, and found that it performs significantly better than all other schemes that we could think of.

The nested factor structure of the model makes it difficult to write down explicitly the corresponding copulas. This illustrates why a formal approach to multivariate copulas is doomed to fail: copulas are not necessarily the natural language in which the specificities of financial markets can be elicited.

There are many avenues of research suggested by the present study. First, it would be interesting to check that other stock markets (EU, UK, JP) lead to the same conclusions, as we believe they will. Second, a joint analysis of the multivariate properties of stock returns and implied volatilities in the corresponding option markets would be highly worthwhile. Third, as we have pointed out above, our model is at this stage purely static, in the sense that we have not specified the dynamics of volatility modes and residuals. Morales et al. [2013] propose a first step toward the integration of cross-sectional dependences and dynamic (scaling) properties of financial time series: they show that multifractality and cross-correlations are much related and suggest a hierarchical construction of stock dependences able to account at the same time for the multi-scaling. This is a very rich subject, since all these objects are expected to show long-range temporal dependence, leverage effects and possibly lagged cross correlations between them. Note that, as we already emphasized, the dominant volatility factor is not the volatility of the market factor. The decomposition of the well known index leverage effect into its various components, and the consequences for index options pricing and VIX, is a very natural issue to investigate first.

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Appendix A. Factor models and PCA

The Principal Components Analysis (PCA) relies on the spectral decomposition of the covariance matrix. It is related to, but different from the logic of the linear factor model. In fact, as we show below, the PCA provides a starting point for the identification of the weights \( \beta_F \) of the factors.

The diagonalization of the sample correlation matrix yields

\[
\frac{1}{T} X'X = \Lambda V V' \]

where \( \Lambda \) is the diagonal matrix of eigenvalues, and the columns of \( V \) are the corresponding eigenvectors. Hence, there always exist (linearly orthogonal) factor series \( \tilde{F} \) such that the return series \( X \) can be
decomposed as

\[ X = \tilde{F} \Lambda \hat{\beta}^\dagger V^\dagger \]  \hspace{1cm} \text{where} \quad \frac{1}{T} \tilde{F}^\dagger \tilde{F} = \mathbb{I}_N.  \tag{17} \]

In order to re-conciliate this decomposition in terms of statistical uncorrelated modes \( \tilde{F} \) with the factors \( F \) of the model, the PCA solution \( \beta \) needs to be identified with Eq. \( \beta \), which we recopy here:

\[ X = F \beta + E. \tag{18} \]

The factors should explain as much as possible of the returns covariances (thus of the portfolio variance), leaving only idiosyncratic residual volatility to be explained by the \( e_i \)'s. Said differently, only those eigenvalues having a significant amplitude should be kept in the identification of the spectral decomposition with the factor model. This procedure is known as “eigenvalue clipping” \cite{NB}. Ordering the eigenvalues in decreasing order, and splitting the first \( M \) \((\text{subscript } M)\) from the last \((N - M)\) \((\text{subscript } N - M)\), it is straightforward to obtain the identification

\[ \beta_{PCA} = \Lambda_{M} \hat{\beta}_{PCA}^\dagger V_{M}^\dagger. \tag{19} \]

At this stage, the series of factors can be formally identified as the first \( M \) spectral modes

\[ F = \tilde{F}_{M} = (XV\Lambda^{-\frac{1}{2}})_{M} \tag{20} \]

such that indeed \( \frac{1}{T}F^\dagger F = \mathbb{I}_M \). However, the corresponding residuals are not orthogonal, since:

\[ E = \tilde{F}_{N-M} \Lambda_{N-M} \hat{\beta}_{N-M}^\dagger V_{N-M} \quad \text{s.t.} \quad \frac{1}{T}E^\dagger E = V_{N-M} \Lambda_{N-M} V_{N-M}^\dagger, \tag{21} \]

and thus cannot be understood as idiosyncrasies of the returns time series.

The PCA can alternatively be thought of as the solution of

\[ \frac{1}{T}X^\dagger X = \beta_{PCA}^\dagger \beta_{PCA} \quad \text{with} \quad \beta_{PCA}^\dagger \beta_{PCA} \text{ diagonal,} \]

where importantly \( \beta_{PCA} \) is full rank, but only the \( M \) modes with largest amplitude \( (\beta_{PCA}^\dagger \beta_{PCA}^\dagger)_{kk} \), \( k = 1, \ldots, M \) are kept after the equation is solved.

The factor model, on the other hand, can be seen as a close alternative to the “eigenvalue clipping” method. It rather attempts to minimize the distance between the off-diagonal elements of the LHS and the RHS with a matrix of weights \( \beta_{F} \) of restricted rank \( M \):

\[ \arg \min \left\| \frac{1}{T}X^\dagger X - \beta_{F}^\dagger \beta_{F} \right\|_{\text{off-diag}}. \tag{22} \]

Numerically, we solve the above equation in the vicinity of the weights \( \beta_{PCA} \) corresponding to the \( M \) largest principal components, and with a quadratic norm. Notice that orthogonality of the lines of \( \beta_{F} \) obtained with this method is not granted, as opposed to the PCA, but what matters is rather the fact that the factors are statistically uncorrelated.

How can one compare the information content of the factors on the one hand, and the Principal Components on the other? The idea is to measure the distance between the eigenspace spanned by the \( M \) largest eigenvectors of the correlation matrix with the \( M \)-dimensional eigenspace spanned by the weights \( \beta_{F} \) of the factor model. A natural measure for this distance was introduced by \cite{Allez2011}, in terms of the \( M \times M \) overlap matrix:

\[ D(M) = \frac{1}{M} \ln \det \hat{\beta}_{F}^\dagger \beta_{PCA}. \tag{23} \]

where \( \hat{\beta}_{F} \) is the orthonormalized set of vectors spanning the same subspace as \( \beta_{F} \). (Note that the \( \beta_{PCA} \) are by construction orthonormal: this is precisely why these weights are hard to interpret directly).

The distance \( D(M) \) for our data set is shown in Fig. \ref{fig:overlap} As expected, \( D(M = 1) \) is very small: both methods identify the most important “mode” or “factor” as the market itself. The overlap between \( \beta_{F} \) and \( \beta_{PCA} \) is in this case 99.9\%. Another trivial limit is \( D(M = N) \), which is zero because the full space is by construction spanned in both cases. Between the two limits, we see that \( D(M) \) always remains very small \(< 4\%\), which means that the information content is different, but similar for the two methods.\footnote{Some “high” values in the bulks may be due to arbitrary permutations in the labeling of the modes. In fact, the PCA prior has a natural ordering (decreasing eigenvalues) but this order of relevant factors is not necessarily conserved in the solution of Eq. \ref{eq: PCA}.} Note that \( D(M) \) is even less than 1\% up to \( M \approx 20 \), i.e. for the most relevant eigenvalues.
APPENDIX B. CALIBRATION OF THE DOMINANT VOLATILITY MODEL

We introduce the Moment Generating Function (MGF) $M_z(p) \equiv \mathbb{E}[\exp(pz)]$. For the $\omega$’s and $\tilde{\omega}$’s, that are assumed to be Gaussian, we have: $M_G(p) = \exp(s^2p^2/2)$. Expanding in powers of $p$, one gets

$$M_D(p) = \exp\left(\frac{1}{2}p^2 + \frac{3}{6}p^3 + \frac{\kappa_0}{24}p^4\right).$$

(24)

It is then convenient to introduce the following ratio of MGF’s:

$$\Phi_0(a, b) = \frac{M_G(a+b)}{M_G(a)M_G(b)},$$

(25)

as well as the Gaussian equivalent $\Phi_G(a, b) = \exp(ab)$. In logarithmic form, $\phi_0(a, b; p) = \frac{1}{p^2} \ln \Phi_0(pa, pb)$ is a polynomial in $p$ when expanding in cumulants. Indeed, with Eq. (24),

$$\phi_0(a, b; p) = ab + \frac{p}{2} \zeta_0(a^2b + ab^2) + \frac{p^2}{12}\kappa_0(2a^3b + 3a^2b^2 + 2ab^3),$$

and $\phi_G(a, b; p) = ab$ is independent of $p$. Then the theoretical prediction for the matrix elements can be computed analytically:

\[
\begin{align*}
\frac{1}{p^2} \ln \frac{\mathbb{E}[|f_k|^p] E[|f_{\ell}|^p]}{\mathbb{E}[|f_k|^p] E[|f_{\ell}|^p]} &= \phi_0(A_{k0}, A_{\ell0}; p) + \left(\gamma(p) + \delta_{k\ell}\right)
\end{align*}
\]

(26a)

\[
\begin{align*}
\frac{1}{p^2} \ln \frac{\mathbb{E}[|f_k|^p] E[|e_i|^p]}{\mathbb{E}[|f_k|^p] E[|e_i|^p]} &= \phi_0(A_{k0}, B_{i0}; p)
\end{align*}
\]

(26b)

\[
\begin{align*}
\frac{1}{p^2} \ln \frac{\mathbb{E}[|e_i|^p] E[|e_j|^p]}{\mathbb{E}[|e_i|^p] E[|e_j|^p]} &= \phi_0(B_{i0}, B_{j0}; p) + \left(\gamma(p) + \delta_{ij}\right)
\end{align*}
\]

(26c)

where

$$\gamma(p) = \frac{1}{p^2} \ln \frac{\mathbb{E}[|e_i|^{2p}]}{\mathbb{E}[|e_i|^p]^2} = \frac{1}{p^2} \ln \left(\sqrt{\pi} \frac{\Gamma(\frac{1}{2} + p)}{\Gamma(\frac{1}{2} + 2p)}\right)$$

is the normalized $2p$-moment of the absolute value of Gaussian variables.

For a Gaussian $\Omega_0$, the correlation matrices defined by Eqs. (26a) and (26c) would be trivially of rank 1, save the diagonal terms. If this was the case, the identification of $A_{0}$ and $B_{0}$ with the first eigenvectors of the corresponding matrices would be straightforward. Non-Gaussianity and specificities on the diagonal terms perturb this identification, but the overall picture is essentially the same story, as we show in Sect. 5 with the calibration results.

The model estimation procedure is as follows (the linear weights $\beta$ are previously estimated). As discussed above, there are $2(N + M + 1)$ parameters to be estimated. Because the equations (26) are coupled through (26b), all parameters should in principle be estimated jointly. The corresponding optimization program would however be computer intensive, and the stability of the solution would not be granted in such a large dimensional space. We proceed stepwise instead, by first estimating the parameters $A_{k0}, s_k, \zeta_0, \kappa_0$ using the fac-fac predictions (26a) and then estimate the remaining parameters $B_{i0}, s_i$ from the res-res and fac-res correlations for consistency. More precisely, our calibration procedure is as follows:

1. Estimate $A_{k0}, s_k$ and the non-Gaussianity parameters from Eq. (26a):

$$\min \left\{ \sum_p \sum_{k, \ell} \left( \frac{1}{p^2} \ln \frac{\langle |F_{k\ell}|^p \rangle}{\mathbb{E}[|F_{k\ell}|^p] \mathbb{E}[|F_{k\ell}|^p]} - \frac{1}{p^2} \ln \frac{\mathbb{E}[|f_k|^p] E[|f_{\ell}|^p]}{\mathbb{E}[|f_k|^p] E[|f_{\ell}|^p]} \right)^2 \right\}$$

(27)

The sum on $p$ runs over eight values between $p = 0.2$ and $p = 2$ and is crucial here to the estimation of the non-Gaussianity parameters, since the loss function is independent of $p$ for Gaussian variables. This amounts to performing a best (joint!) quadratic fit of the curves similar to Fig. 3 for each period.

2. Estimate $B_{i0}$ from Eq. (26b):

$$\min \left\{ \sum_{k, i} \left( \frac{1}{p^2} \ln \frac{\langle |F_{k\ell}|^p \rangle}{\mathbb{E}[|F_{k\ell}|^p] \mathbb{E}[|F_{k\ell}|^p]} - \frac{1}{p^2} \ln \frac{\mathbb{E}[|f_k|^p] E[|e_i|^p]}{\mathbb{E}[|f_k|^p] E[|e_i|^p]} \right)^2 \right\}$$

(28a)
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or jointly with \( \tilde{s}_i \) from Eq. (26c), as the vector solution of

\[
\min \left\{ \sum_{i,j} \left( \frac{1}{p^2} \ln \frac{\langle |E_{i}E_{j}|^p \rangle}{\langle |E_{i}|^p \rangle \langle |E_{j}|^p \rangle} - \frac{1}{p^2} \ln \frac{\mathbb{E}[|\epsilon_i|^p|\epsilon_j|^p]}{\mathbb{E}[|\epsilon_i|^p] \mathbb{E}[|\epsilon_j|^p]} \right)^2 \right\}
\]

(28b)

(here it is too intensive to calculate the optimum in the \( N \)-dimensional space for all values of \( p \) so we take a single value, typically \( p = 1 \) if we intend to reproduce best absolute correlations, or \( p = 2 \) if we favor quadratic correlations).

The convergence is ensured by starting close to the solution, namely taking as prior the first eigenvector of the corresponding matrices.

Appendix C. Test of the Out-of-Sample Performance of Correlation Models

The protocol proposed by Potters and Bouchaud (2009) to compare the out-of-sample risk of different correlation models is as follows:

1. The model is calibrated in windows of \( T^\text{IS} = 2N = 524 \) days. An optimal portfolio is built and a corresponding risk measure is computed over the window used for estimation: this is the In-sample risk. We consider below two kinds of risks corresponding to two different portfolios:

   (i) the quadratic risk of a basket of returns, that will assess the quality of the linear elements of the model; and

   (ii) the quadratic risk of a basket of (centered and normalized) absolute returns, that will assess the quality of the volatility description of the model.

2. The same risk measures are computed Out-of-sample on a small period of \( T^\text{OS} = 59 \) days (three months) following the estimation period.

3. The sliding lags are chosen so that the control samples are non-overlapping, i.e. at dates \( \tau = T^\text{IS} + n \times T^\text{OS} + 1, n = 0, 1, 2, \ldots \). Sliding windows will be indexed with parenthesis notation \( (\tau) \), in order to avoid confusion with regular time stamps \( t \) of the running dates.

We then build a portfolio of assets \( y_i \) knowing their historical time series \( Y_{ti} \), which can be returns in the standard case, but also absolute returns or squared returns when one has non-linear assets in mind (such as options, for example).

For a given covariance matrix \( \rho_{ij} = \text{cov}(y_i, y_j) \), optimal portfolio weights can be computed in the sense of Markowitz:

\[
w^*(\tau) = \frac{\rho^{-1}g(\tau)}{g(\tau)^T \rho^{-1}g(\tau)}
\]

(29)

where we consider an omniscient stationary predictor of returns

\[
g(\tau) = \frac{Y_{\tau i}}{\sqrt{\frac{1}{N} \sum_j Y_{\tau j}^2}}
\]

(30)

and a unit total gain \( \mathcal{G} \equiv g^T w^* = 1 \). This means that the in-sample/out-of-sample test procedure applied below is intended to measure only risk and not the risk-return trade-off (Sharp ratio) as is usual e.g. when back-testing financial strategies. Indeed what we ultimately want to conclude is whether our model of stock returns allows to have a better view of dependences and thus to better diversify away the risk (since we work with normalized returns, we are not concerned with individual variances but only care for dependences).

Quadratic risk is essentially a measure of expected small fluctuations of the portfolio value:

\[
\mathcal{R}^2(\tau) = \frac{1}{T'} \sum_{t'} \left\{ \frac{1}{N} \sum_{i=1}^N \left[ Y_{t' i} w^*_i(\tau) \sigma^\text{IS}_i(\tau) \right]^2 \right\}^2
\]

(31)

where, for convenience, the asset returns are normalized by a rolling in-sample estimate of their volatility \( \sigma^\text{IS}(\tau) \) — although the returns have been normalized over the whole period, they may not be close to unit-variance in-sample because of low-frequency regime switches in the volatility. This risk is computed both in-sample (in which case \( T' = T^\text{IS} \) and \( t' = \tau - T', \ldots, \tau - 1 \)) and out of sample (\( T' = T^\text{OS} \) and \( t' = \tau + 1, \ldots, \tau + T' \)), for different input correlation matrices in Eq. (29).

References


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Figure 9. **Left:** calibrated vs sample quadratic correlations. **Middle:** sample quadratic correlations vs sample linear correlations; **Right:** calibrated quadratic correlations vs calibrated linear correlations. Two benchmark curves are added in red: the Gaussian case (lower curve) and the Student case with $\nu = 5$ d.o.f. (upper curve).
Figure 10. 2000-2004. diagonal (et) and anti-diagonal (trif) of the copula as in Eq. (14).

(a) Empirical

(b) Calibrated

(c) Empirical

(d) Calibrated

\( \rho \text{ moy} = 0.822 \) for the range of \( ho \) between 0.8 and 1.

\( \rho \text{ moy} = 0.658 \) for the range of \( ho \) between 0.6 and 0.8.

\( \rho \text{ moy} = 0.504 \) for the range of \( ho \) between 0.45 and 0.6.

\( \rho \text{ moy} = 0.391 \) for the range of \( ho \) between 0.35 and 0.45.

\( \rho \text{ moy} = 0.295 \) for the range of \( ho \) between 0.25 and 0.35.

\( \rho \text{ moy} = 0.02 \) for the range of \( ho \) between −1 and 0.05.

\( \rho \text{ moy} = 0.016 \) for the range of \( ho \) between −1 and 0.05.

\( \rho \text{ moy} = 0.649 \) for the range of \( ho \) between 0.6 and 0.8.

\( \rho \text{ moy} = 0.504 \) for the range of \( ho \) between 0.45 and 0.6.
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Figure 11. Two volatility modes $\Omega_0$ and $\Omega_1$. $M = 10$, 2000–2004

(c) Index amplitudes reproduced with two volatility drivers $\Omega_0$ and $\Omega_1$, to be compared with Fig. S(a)
Figure 12. Linear correlations: Out-of-sample risk vs In-sample risk, defined in Eq. (31) and averaged over sliding windows in 2000–2009, for three cleaning schemes: eigenvalue clipping (red circles) and calibrated multi-factor model (blue crosses), both with $M$ linear factors, as well as Ledoit-Wolf shrinkage (cyan line). Inset: the Out-of-sample risk is lowered by more than 5% with respect to “Clipping”, when $M \approx 24$. 
Figure 13. Absolute correlations: Out-of-sample risk vs In-sample risk, defined in Eq. (31) and averaged over sliding windows in 2000–2009, for three cleaning schemes: eigenvalue clipping of $M$ modes of quadratic correlation (red circles), Gaussian multi-factor (magenta triangles) and calibrated multi-factor model (blue crosses), both with $M$ linear factors. The role of volatility dependences is elicited by the better performance of the non-Gaussian multi-factor level over the Gaussian multi-factor cleaning scheme: the relative risk difference (16) is shown in the inset, and is always negative.

Figure 14. Overlap $D(M)$ between the spaces spanned by $\hat{\beta}_F$ and $\beta_{PCA}$ when $M$ factors are retained, see definition in Eq. (23).