

Eigenvectors dynamic and local density of states under free addition

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January 22, 2013

Abstract

We investigate the evolution of a given eigenvector of a symmetric (deterministic or random) matrix under the addition of a matrix in the Gaussian orthogonal ensemble. We quantify the overlap between this single vector with the eigenvectors of the initial matrix and identify precisely a “Cauchy-flight” regime. In particular, we compute the local density of this vector in the eigenvalues space of the initial matrix. Our results are obtained in a non perturbative setting and are derived using the idea of Ledoit and Péché in [11]. Finally, we revisit our former results on the eigenspace dynamics giving a robust derivation of a result obtained in [1] in a semi-perturbative regime.

1 Introduction

The dynamics of eigenvalues induced by the addition of free random matrices in the Gaussian orthogonal ensemble has been first studied by Dyson in his 1962 paper [8]. The movement of the eigenvalues is characterized in terms of a stochastic differential system, the so called *Dyson Brownian motion*. The eigenvalues evolve as particles of a Coulomb gas with electrostatic repulsion, confined in a quadratic potential and subject to a thermal noise. In the limit of large dimensions, the evolution of the spectral density has also been studied in [13] (see also [6, 5] and [7, 2] for related models).

For the eigenvectors, their evolution in finite dimension is also given by a stochastic differential system which depend on the non colliding trajectories of the eigenvalues (see [3]). In this paper, we are interested in quantifying the evolution of the eigenvectors in the limit of large dimension. Our approach uses the idea of [11] who

introduced a very interesting quantity (see Eq. (5.1) below) for the study of eigenvectors. This enables us to compute the local density of a given state (eigenvector) of the matrix after the addition of the free Gaussian matrix, in the eigenvalues space of the initial matrix.

The paper is organized as follows. In section 2, we define the model and give the main notations. We enunciate our main result (Theorem 3.2) in section 3 concerning the convergence of the quantity (5.1) introduced in [11]. In the following section 4, we are interested in the local density of states under free addition. We find that in a particular regime, the eigenvalue dynamics can be precisely described as a ‘‘Cauchy flight’’. We also check numerically our results in the case of a initial random matrix in the Gaussian orthogonal ensemble. We then revisit in section 4.4 the main result of [1] on the dynamics of eigenspace under free addition, and prove that it is indeed exact beyond the perturbative regime. Finally, section 5 is devoted to the proof of Theorem 3.2.

Acknowledgments We are grateful to Yan Fyodorov, Denis Ullmo and Matthieu Wyart who asked questions about [1] which led to the work presented here. We would like to thank also Sandrine P  ch   for interesting discussions.

2 Definition and main notations

Let A be a symmetric deterministic $N \times N$ matrix. The matrix A is diagonalizable in an orthonormal basis of \mathbb{R}^N and its eigenvalues will be denoted in increasing order as $a_1 \leq a_2 \leq \dots \leq a_N$. We will suppose that the a_i are allocated smoothly as $a_i = a(\frac{i}{N})$ where $a(\cdot)$ is a ‘‘nice’’ differentiable and strictly increasing function defined on the interval $[0; 1]$.

Let $(H_t)_{t \geq 0}$ be a symmetric Brownian motion, i.e. a symmetric diffusive matrix process constructed from a family of independent real Brownian motions $B_{ij}(t)$, $1 \leq i \leq j \leq N$ as follows

$$H_t(ij) = \begin{cases} \sqrt{\frac{1}{N}} B_{ij}(t) & \text{if } i < j, \\ \sqrt{\frac{2}{N}} B_{ii}(t) & \text{if } i = j. \end{cases}$$

Note that the process H_t is defined such that it is rotationally invariant at all time, in the sense that for all O in the orthogonal group \mathcal{O}_N , the conjugate matrix OH_tO^\dagger has the same law as the matrix H_t .

Now we define the matrix M_t such that for all $t \geq 0$

$$M_t = A + H_t. \tag{2.1}$$

The eigenvalues of the matrix M_t will be denoted in increasing order as $\lambda_1^t \leq \dots \leq \lambda_N^t$.

The aim of this note is to quantify the relationship between the eigenvectors of the matrix M_t with the eigenvectors of the initial matrix $M_0 = A$. In particular, we consider one given eigenvector of the matrix M_t denoted as $|\psi_i^t\rangle$ and we want

to compute, in the large limit of large dimension N , the overlaps of this vector $|\psi_i^t\rangle$ with the eigenvectors of A denoted in the following as $|\phi_1\rangle, \dots, |\phi_N\rangle$ (which form an orthonormal basis). As the matrix H_t is rotationally invariant for all t , we can suppose with no loss of generality that the matrix A is *diagonal*.

3 Results

If t is very small, perturbation theory for the eigenvectors of the matrix M_t defined in (2.1) gives an approximation of the vector $|\psi_i^t\rangle$ in terms of the small matrix H_t and of the eigenvectors of the matrix A . More precisely, if t is small compared to the level spacings of the matrix A , perturbation theory leads to (see appendix A for a reminder of this computation)

$$|\psi_i^t\rangle = \left(1 - \frac{1}{2} \sum_{j \neq i} \frac{\langle \psi_i^t | H_t | \phi_j \rangle^2}{(a_i - a_j)^2} \right) |\phi_i\rangle + \sum_{j \neq i} \frac{\langle \psi_i^t | H_t | \phi_j \rangle}{a_i - a_j} |\phi_j\rangle + |\varepsilon_t\rangle, \quad (3.1)$$

where $|\varepsilon_t\rangle$ is a vector such that $\langle \phi_i | \varepsilon_t \rangle = o(t)$ and $\langle \phi_j | \varepsilon_t \rangle = o(\sqrt{t})$ for $j \neq i$. This expansion (3.1) can be rewritten, taking a further expectation, in terms of the overlap between $|\psi_i^t\rangle$ and $|\phi_j\rangle$ for $i \neq j$ as

$$\mathbb{E} [\langle \psi_i^t | \phi_j \rangle^2] = \frac{t}{N} \frac{1}{(a_i - a_j)^2} + o(t), \quad (3.2)$$

and for $i = j$,

$$\mathbb{E} [\langle \psi_i^t | \phi_i \rangle^2] = 1 - \frac{t}{N} \sum_{j \neq i} \frac{1}{(a_i - a_j)^2} + o(t). \quad (3.3)$$

The computations leading to equation (3.1) are *a priori* only valid for values of t much smaller than the spacings between the consecutive eigenvalues $a_i - a_{i-1}$, so that the correction terms in the perturbation equation (3.1) remains very small compared to the leading term of order 1. Note that in our setting and for large N , $a_i - a_{i-1}$ is of order $1/N$, so that t has to be much smaller than $1/N$.

The purpose at stake here is to handle the case of larger values of t in the large N limit. For this eigenvector dynamics problem, there are in fact three regime for the values of t and N :

- The first regime is the *perturbative regime* where $t \ll 1/N$, for which the perturbation theory applies to give the approximation (3.1). This regime is well known and has been studied in great details in random matrix theory and in the context of quantum mechanics.
- The second regime is what we will call *semi-perturbative* and concerns small values of t compared to 1 but not necessarily small compared to the level spacings of order $1/N$. It includes values such that $1/N \ll t \ll 1$.

- The third regime is *non perturbative*: N is very large while t is of order 1.

The question we ask is: How do formulas (3.2) and (3.3) get modified in the second and third regime in the large N limit? Because the family $\{\phi_j, 1 \leq j \leq N\}$ forms an orthonormal basis of \mathbb{R}^N , we have

$$\sum_{j=1}^N \langle \psi_i^t | \phi_j \rangle^2 = 1.$$

We will therefore investigate the convergence in the large N limit of $N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2]$ for $i \neq j$. Those quantities are in fact related to the mean *local density of state* $|\psi_i^t\rangle$ which is the probability measure ν_i defined on the A -eigenvalues space as

$$\nu_i = \sum_{j=1}^N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2] \delta(a_j)$$

where $\delta(x)$ is the Dirac measure in x . In other words, the aim of the note is to compute the local density ν_i of state $|\psi_i^t\rangle$ in the large N limit.

The interesting quantity for our purpose is the bivariate cumulative distribution function Φ associated to the weights $N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2]$ defined for $\lambda, \alpha \in \mathbb{R}$ by

$$\Phi_N(\lambda, \alpha) = \frac{1}{N} \sum_{i,j=1}^N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2] \mathbf{1}_{\{\lambda_i^t \leq \lambda\}} \mathbf{1}_{\{a_j \leq \alpha\}}.$$

Note that this function Φ has indeed the properties of a bivariate cumulative distribution function since it is right continuous with left-hand limits and is nondecreasing in each of its argument and that it satisfies $\lim_{\lambda \rightarrow -\infty, a \rightarrow -\infty} \Phi(\lambda, a) = 0$ and

$$\lim_{\lambda \rightarrow +\infty, a \rightarrow +\infty} \Phi(\lambda, a) = 1.$$

Before presenting our results on the convergence in the large N limit of the bivariate cumulative distribution $\Phi_N(\lambda, \alpha)$ which will directly lead us to asymptotic estimates for the overlaps $N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2]$ for $i \neq j$ and for the local density of states, let us first recall a result on the convergence of the empirical eigenvalue distribution of the matrix M_t defined in (2.1) due to Shlyakhtenko [14] (see also [12, 19] for similar results). Recall the definition of the Stieltjes transform $G_\mu(z)$ of a probability measure μ on \mathbb{R} defined for all $z \in \mathbb{C} \setminus \mathbb{R}$ as $G_\mu(z) = \int_{\mathbb{R}} \frac{\mu(dx)}{x-z}$.

Theorem 3.1 (Shlyakhtenko, [14]). *The empirical eigenvalue distribution of the matrix M_t converges weakly almost surely when $N \rightarrow \infty$ to a probability measure μ_t whose Stieltjes transform satisfies*

$$G_{\mu_t}(z) = \int_0^1 G_t(z, x) dx$$

where $G_t(z, x)$ is defined uniquely for any $z \in \mathbb{C} \setminus \mathbb{R}$ by the fact that it is analytic on $\mathbb{C} \setminus \mathbb{R}$, maps the upper half plane \mathbb{C}^+ into the lower one \mathbb{C}^- , and satisfies the relation for all $z \in \mathbb{C} \setminus \mathbb{R}$ and $x \in [0; 1]$

$$G_t(z, x) = \frac{1}{a(x) - z - t \int_0^1 G_t(z, y) dy}. \quad (3.4)$$

Furthermore, setting $R_t^N(z) = (M_t - zI)^{-1}$, the complex random measure on the interval $[0; 1]$ defined as

$$\sigma_t^N = \frac{1}{N} \sum_{i=1}^N \mathbb{E}[R_t^N(z)_{ii}] \delta\left(\frac{i}{N}\right) \quad (3.5)$$

converges weakly almost surely to the complex measure with density $G(z, x)dx$.

The proof of this theorem goes through the Schur complement formula, the fixed point theorem and classical properties of analytical functions.

Finally, using the work of Biane [6], we know that the limiting spectral distribution $\mu_t(d\lambda)$ admits a smooth density $\rho_t(\lambda)$ with respect to Lebesgue measure. We further stress the convergence of the Stieltjes transform near the real axis:

$$\lim_{\eta \rightarrow 0_+} G_{\mu_t}(\lambda + i\eta) = H_{\rho_t}(\lambda) + i\pi\rho_t(\lambda)$$

where H_{ρ_t} is the Hilbert transform of the probability density ρ_t .

We are now ready to state our main result on the convergence of the bivariate cumulative distribution $\Phi_N(\lambda, \alpha)$.

Theorem 3.2. *When $N \rightarrow +\infty$, we have the following convergence*

$$\Phi_N(\lambda, \alpha) \longrightarrow \Phi(\lambda, \alpha) = \int_{-\infty}^{\lambda} d\xi \rho_t(\xi) \int_{-\infty}^{\alpha} \frac{dx}{a'(a^{-1}(x))} \frac{t}{(x - \xi - t H_{\rho_t}(\xi))^2 + t^2 \pi^2 \rho_t(\xi)^2}.$$

We will see in the following section that Theorem 3.2 permits to compute the asymptotic overlaps $N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2]$ of any eigenvector $|\psi_i^t\rangle$ of the matrix M_t at time t with the eigenvectors of the initial matrix $M_0 = A$ for any time t . A former heuristic attempt to compute those overlaps already appeared in [17], with a different result (see below). Theorem 3.2 also permits to compute the mean local density ν_i in the A -eigenvalues space of the state $|\psi_i^t\rangle$ at time t . In addition we will explain how Theorem 3.2 enables to extend the domain of validity of our former result on the eigenspace dynamics under free addition obtained in [1].

Theorem 3.2 is similar to [11, Theorem 1.3]. In [11], the authors investigate the relationship between the eigenvectors of the population covariance matrix with those of the empirical (or sample) covariance matrix.

One remarkable feature of Theorem 3.2 is that it quantifies the relationship between the eigenvectors of the initial matrix $M_0 = A$ and the eigenvectors of the

matrix M_t , even in the *non perturbative* third regime described above (i.e. it holds for any time $t > 0$). In [11], the sample covariance matrix is an additive perturbation of the population covariance matrix. The authors of [11] thus work in the second *semi-perturbative* regime described above (note that there is not a third regime in their case though).

Our proof of Theorem of 3.2 uses the idea used in [11] to quantify the relationship between sample and population eigenvectors.

4 Applications of Theorem 3.2

The present section contains physical arguments and is not fully rigorous mathematically.

4.1 Perturbation theory revisited

Theorem 3.2 enables to compute the overlaps of the vector $|\psi_i^t\rangle$ with the initial ($t = 0$) eigenvectors $|\phi_j\rangle$ for $j \neq i$ of the matrix $M_0 = A$. We can deduce from Theorem 3.2 that, for $N \rightarrow \infty$, we have

$$\mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2] = \frac{1}{N} \frac{t}{(a_j - \lambda_i^t - tH_{\rho_t}(\lambda_i^t))^2 + t^2\pi^2\rho_t(\lambda_i^t)^2} + o\left(\frac{1}{N}\right). \quad (4.1)$$

As mentioned before, formula (4.1) is valid for all values of t and N in the third regime described above (i.e. for large N and any $t > 0$).

For $t \ll 1$ and for eigenvalues a_i, a_j separated by a macroscopic spacing (i.e. such that $a_j - a_i$ does not vanish for large N), we have $\lambda_i^t \approx a_i$ and formula (4.1) rewrites as

$$\mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2] = \frac{t}{N} \frac{1}{(a_j - a_i)^2} + o\left(\frac{1}{N}\right). \quad (4.2)$$

For such i, j , (4.2) extends the perturbation equation 3.2 (which is valid only for $t \ll 1/N$) to values of t much smaller than 1 but not necessarily negligible compared to $1/N$.

4.2 Local density of state

In other words, Theorem 3.2 gives the local density of states in the large N limit. The limiting mean local density of states $|\psi_i^t\rangle$ in α is

$$\nu_i(\alpha) = \frac{1}{a'(\alpha^{-1}(\alpha))} \frac{t}{(\alpha - \lambda_i - tH_{\rho_t}(\lambda_i))^2 + t^2\pi^2\rho_t(\lambda_i)^2}. \quad (4.3)$$

This last formula (4.3) defines a probability density on \mathbb{R} although it is not trivial to check that its integral over α is indeed 1 (the probability measure ρ_t and its Hilbert transform depend on the function $a(x)$ so that ν_i is indeed a probability density function).

4.3 The case of Dyson Brownian motion

In this section, we apply our results to the particular case in which the initial matrix A is a random matrix of the Gaussian orthogonal ensemble (GOE). As before, we will work in the basis of the eigenvectors of A , so that the matrix A is diagonal with eigenvalues in increasing order: $A = \text{Diag}(a_1, \dots, a_N)$ with $a_1 < \dots < a_N$. The spectrum of A is random in this case, and the assumption we made to derive our results on the smooth allocation of its spectrum is *not* satisfied here. We will check numerically that our results hold also in this case though.

Using Wigner's semi circle law, we can conjecture that the spectrum of A should converge to the continuous and strictly function $a : [0; 1] \rightarrow [-2; 2]$ such that, for all $x \in [0; 1]$,

$$\frac{1}{2\pi} \int_{-2}^{a(x)} \sqrt{4 - t^2} dt = x.$$

The matrix M_t is also a GOE matrix and its limiting empirical eigenvalue density is given by the semi-circle law

$$\rho_t(\lambda) = \frac{1}{2\pi(1+t)} \sqrt{4(1+t) - \lambda^2}. \quad (4.4)$$

It is also known (see e.g. [3]) that the associated Stieltjes transform $G_{\mu_t}(z)$ satisfies

$$\lim_{\eta \rightarrow 0_+} G_{\mu_t}(\lambda + i\eta) = -\frac{\lambda}{2(1+t)} + i\pi\rho_t(\lambda).$$

Hence, Theorem 3.2 gives the following expression for the asymptotic overlaps for $i \neq j$:

$$N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2] = \frac{t}{(a_j - \lambda_i^t)^2 + \frac{t}{1+t} \lambda_i^t (a_j - \lambda_i^t) + \frac{t^2}{1+t}} + o(1). \quad (4.5)$$

We have checked those asymptotic expressions, using numerical simulations, in the present context of GOE matrices of dimension $N = 400$ with $t = 1$. The agreement is excellent: see Fig. 1 and 2.

In other words, the local density ν_i of state $|\psi_i^t\rangle$ in the limit of large matrices is the probability measure with density in the A -eigenvalues space

$$\nu_i(\alpha) = \frac{\sqrt{4 - \alpha^2}}{2\pi} \frac{t}{(a_j - \lambda_i^t)^2 + \frac{t}{1+t} \lambda_i^t (a_j - \lambda_i^t) + \frac{t^2}{1+t}}.$$

When $t \rightarrow +\infty$, the information about the initial state is finally lost

$$\lim_{t \rightarrow +\infty} \lim_{N \rightarrow +\infty} N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2] = 1.$$

The vector $|\psi_i^t\rangle$ has uniform overlaps with the initial eigenvectors $|\phi_j\rangle$ as expected.

In the limit $t \ll 1$, Eq. (4.5) can be simplified to read:

$$N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2] \approx \frac{t}{(a_j - \lambda_i^t)^2 + t^2}, \quad (4.6)$$

which describes a ‘‘Cauchy flight’’ in eigenvalue space. This makes more precise a statement made in [17, 18, 1] in the context of an extreme nonadiabatic evolution of a quantum system: the energy is not diffusive but rather performs a Cauchy Flight.

In fact, if the evolution of the system is such that the elements of the random GOE matrix M_t have a fixed variance, the i -th eigenvalue of M_t is expected to be time independent in the large N limit, i.e. $\lambda_i^t \approx a_i$. In this case, Eq. (4.6) corresponds (up to simple modifications) to Eq. (4.11) of [17], with the correspondence $\Delta E = a_j - a_i$. However, the correspondence for longer ‘‘times’’ t must take into account that with our normalization, the semi-circle spectrum itself broadens with time, as given by Eq. (4.4).

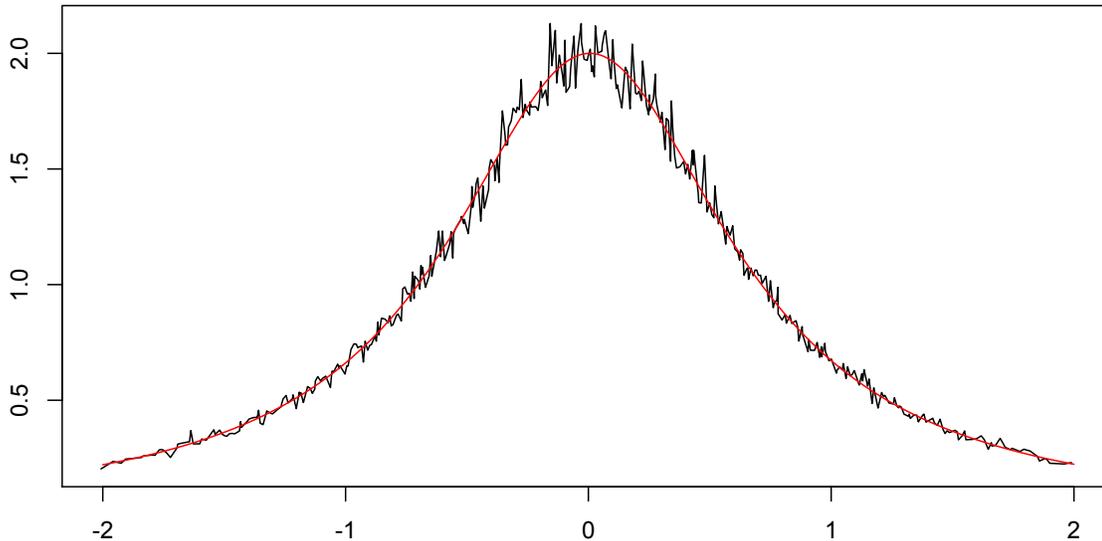


Figure 1: The black curve is a plot of $N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2]$, computed empirically with 1000 samples, as a function of the eigenvalues a_j corresponding to $|\phi_j\rangle$, for $N = 400$, $t = 1$ and $i = 200$. The 200 th eigenvalue of A is approximately equal to 0, it is therefore natural to observe the highest value of this curve at this point. The red curve is the theoretical prediction displayed in Eq. (4.5).

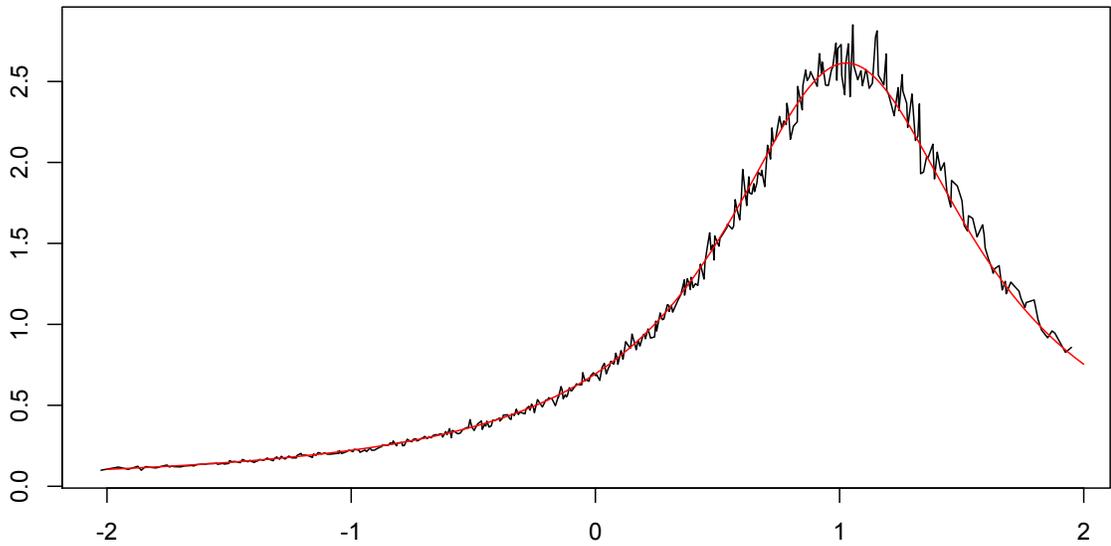


Figure 2: The black curve is a plot of $N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2]$, computed empirically with 1000 samples, as a function of the eigenvalues a_j corresponding to $|\phi_j\rangle$, for $N = 400$, $t = 1$ and $i = 320$. The 320 th eigenvalue of A is approximately equal to 0.983, it is therefore natural to observe the highest value of this curve near this point. The red curve is the theoretical prediction displayed in Eq. (4.5).

4.4 Eigenspace stability

In [1], we investigated the stability of eigenspaces associated to a GOE matrix A when a small GOE matrix H_t is added. Let us briefly recall the context and main notations of [1].

Our idea was to study, in the large N limit, the stability of a whole subspace V_0 (instead of a single eigenvector as above) spanned by a set of consecutive initial eigenvectors $|\phi_k\rangle$ associated to eigenvalues a_k contained in a certain interval $[\gamma_-; \gamma_+]$ of the Wigner semicircle support $[-2; 2]$. We then asked the following question: How should one choose a “larger” subspace V_1^t spanned by a subset of eigenvectors $|\psi_k^t\rangle$ at time t which would contain the initial subspace V_0 up to a small error? To answer this question, we introduced a margin of width δ and the subspace V_1^t generated by the set of eigenvectors $|\psi_k^t\rangle$ associated to eigenvalues λ_k^t lying in the interval $[\gamma_- - \delta; \gamma_+ + \delta]$. We then considered the rectangular matrix of overlaps G_t with entries

$$G_t(ij) := \langle \psi_i^t | \phi_j \rangle.$$

In this setting, the matrix G_t has dimensions $Q \times P$ with

$$P = N \int_{\gamma_-}^{\gamma_+} \rho_0, \quad Q = N \int_{\gamma_- - \delta}^{\gamma_+ + \delta} \rho_t,$$

where ρ_0 is the Wigner semicircle eigenvalues density of the initial matrix A . The labels i and j and the vectors $|\psi_i^t\rangle$ and $|\phi_j\rangle$ are respectively indexed by the eigenvalues (in increasing order) λ_i^t and a_j .

The P nonzero singular values $1 \geq s_1 \geq s_2 \geq \dots \geq s_P \geq 0$ of the matrix G_t contain full information about the overlap between the two spaces V_0 and V_1^t . For example, the largest singular value s_1 indicates that there is a certain linear combination of the Q eigenvectors at time t that has a scalar product s_1 with a certain linear combination of the P initial eigenvectors. If $s_P = 1$, then the initial subspace is entirely spanned by the perturbed subspace. If on the contrary $s_1 \ll 1$, it means that the initial and perturbed eigenspaces are nearly orthogonal to one another since even the largest possible overlap between any linear combination of the original and perturbed eigenvectors is very small. A good measure of what can be called an overlap distance $D(V_0, V_1^t)$ between the two spaces V_0 and V_1 is provided by the average of the logarithm of the singular values:

$$D(V_0, V_1^t) = -\frac{1}{P} \sum_{k=1}^P \ln(s_k).$$

This overlap distance D and the overlap matrix G_t already appeared in the literature on the “Anderson orthogonality catastrophe” (see e.g. [4, 10]).

Using perturbation theory, we showed in [1] that this overlap distance $D(V_0, V_1^t)$, for $N \gg 1$ and in $Nt \ll 1$, converges to:

$$\mathbb{E}[D(V_0, V_1^t)] = \frac{t}{2 \int_{\gamma_-}^{\gamma_+} \rho_0} \int_{\gamma_-}^{\gamma_+} dx \int_{y \notin [\gamma_- - \delta, \gamma_+ + \delta]} dy \frac{\rho_0(x)\rho_0(y)}{(x-y)^2} + o(t). \quad (4.7)$$

We conjectured that this formula also holds in the semi-perturbative regime $N \gg 1$ and $t \ll 1$ but $Nt \gg 1$. This conjecture was supported by convincing numerical evidence [1]. Unfortunately we were unable at the time to find analytical arguments to sustain our claim in this semi-perturbative regime.

We will use our new results obtained here to fill this gap and prove that Eq. (4.7) is indeed correct in the whole regime for $t \ll 1$ (independent of N) in the limit $N \rightarrow +\infty$.

Let us first remark that $D(V_0, V_1^t) = -\ln(\det(G_t^\dagger G_t))/(2P)$ where G_t^\dagger is the Hermitian conjugate of G_t . We thus compute the entries of the matrix $G_t^\dagger G_t$. For all $a_i \in [\gamma_-; \gamma_+]$, we have

$$\begin{aligned} (G_t^\dagger G_t)_{ii} &= \sum_{\lambda_k^t \in [\gamma_- - \delta; \gamma_+ + \delta]} \langle \psi_k^t | \phi_i \rangle^2 \\ &= \langle \psi_i^t | \phi_i \rangle^2 + \sum_{k \neq i: \lambda_k^t \in [\gamma_- - \delta; \gamma_+ + \delta]} \langle \psi_k^t | \phi_i \rangle^2 \\ &= 1 - \sum_{k \neq i} \langle \psi_k^t | \phi_i \rangle^2 + \sum_{k \neq i: \lambda_k^t \in [\gamma_- - \delta; \gamma_+ + \delta]} \langle \psi_k^t | \phi_i \rangle^2 \\ &= 1 - \sum_{\lambda_k^t \notin [\gamma_- - \delta; \gamma_+ + \delta]} \langle \psi_k^t | \phi_i \rangle^2. \end{aligned}$$

Using the previous result (4.5) obtained in the previous subsection, we see that for values of $t \ll 1$ and for large N , we have

$$\begin{aligned} \mathbb{E}[(G_t^\dagger G_t)_{ii}] &= 1 - \frac{t}{N} \sum_{a_k \notin [\gamma_- - \delta; \gamma_+ + \delta]} \frac{1}{(a_i - a_k)^2} + o\left(\frac{1}{N}\right) + o(t) \\ &\sim_{N \rightarrow \infty} 1 - t \int_{y \notin [\gamma_- - \delta; \gamma_+ + \delta]} dy \frac{\rho_0(y)}{(a_i - y)^2} + o(t) \end{aligned} \quad (4.8)$$

where we have used the fact that for small values of t , the eigenvalues satisfy $\lambda_k^t \sim \lambda_k^0 = a_k$ for the last line.

The non diagonal elements, i.e. indexed by $a_i \neq a_j \in [\gamma_-; \gamma_+]$, can also be computed as

$$\begin{aligned} (G_t^\dagger G_t)_{ij} &= \sum_{\lambda_k^t \in [\gamma_- - \delta; \gamma_+ + \delta]} \langle \psi_k^t | \phi_i \rangle \langle \psi_k^t | \phi_j \rangle \\ &= - \sum_{\lambda_k^t \notin [\gamma_- - \delta; \gamma_+ + \delta]} \langle \psi_k^t | \phi_i \rangle \langle \psi_k^t | \phi_j \rangle \end{aligned}$$

where, in the second line, we have used the orthogonality of $|\phi_i\rangle$ and $|\phi_j\rangle$ which implies that $\sum_k \langle \psi_k^t | \phi_i \rangle \langle \psi_k^t | \phi_j \rangle = 0$. The expectations of those terms can thus be estimated for small values of t and large N , via the Cauchy-Schwarz inequality, as

$$\begin{aligned} \mathbb{E}[(G_t^\dagger G_t)_{ij}] &\leq \sum_{a_k \notin [\gamma_- - \delta; \gamma_+ + \delta]} \mathbb{E}[\langle \psi_k^t | \phi_i \rangle]^{1/2} \mathbb{E}[\langle \psi_k^t | \phi_j \rangle]^{1/2} \\ &\sim_{N \rightarrow +\infty} \frac{t}{N} \sum_{a_k \notin [\gamma_- - \delta; \gamma_+ + \delta]} \frac{1}{(a_i - a_k)(a_j - a_k)} + o\left(\frac{1}{N}\right) + o(t) \end{aligned} \quad (4.9)$$

$$\sim_{N \rightarrow +\infty} t \int_{y \notin [\gamma_- - \delta; \gamma_+ + \delta]} dy \frac{\rho_0(y)}{(a_i - y)(a_j - y)} + o(t). \quad (4.10)$$

Note that in both equations (4.8) and (4.10), we have $a_i, a_j \in [\gamma_-; \gamma_+]$ and $a_k \notin [\gamma_- - \delta; \gamma_+ + \delta]$, so that a_i (resp. a_j) and a_k remain at macroscopic distance $> \delta$ and formula (4.2) applies. Besides the integrals in (4.8) and (4.10) are perfectly well defined due to the introduction of the margin $\delta > 0$.

Thus, for $t \ll 1$ and large N , the determinant of $G_t^\dagger G_t$ can be approximated, in the large N limit, to leading order in t as the product of the diagonal terms (the other contribution are negligible compared to t for small t). We thus have, doing a further linearization in the limit of small t ,

$$\begin{aligned} -\frac{1}{2P} \mathbb{E}[\ln(\det(G_t^\dagger G_t))] &= \frac{t}{P} \sum_{a_i \in [\gamma_-; \gamma_+]} \int_{y \notin [\gamma_- - \delta; \gamma_+ + \delta]} dy \frac{\rho_0(y)}{(a_i - y)^2} + o\left(\frac{1}{N}\right) + o(t) \\ &\sim_{N \rightarrow +\infty} \frac{t}{\int_{\gamma_-}^{\gamma_+} \rho_0} \int_{\gamma_-}^{\gamma_+} dx \int_{y \notin [\gamma_- - \delta; \gamma_+ + \delta]} dy \frac{\rho_0(x)\rho_0(y)}{(x - y)^2} + o(t). \end{aligned}$$

This is our proof that (4.7) is valid in the second semi-perturbative regime.

The reader may wonder how to extend formula (4.7) in the non perturbative regime, i.e. for arbitrary values of t . This question is clearly more difficult as one would need to understand the convergence of the non diagonal terms of the matrix $G_t^\dagger G_t$ in the large N limit which are no longer negligible in the determinant expansion.

5 Proof of Theorem 3.2

Following the idea of [11], we introduce the following quantity, defined for $z \in \mathbb{C} \setminus \mathbb{R}$, as

$$\begin{aligned} \Theta_N^g(z) &= \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i^t - z} \sum_{j=1}^N \mathbb{E}[\langle \psi_i^t | \phi_j \rangle^2] g(a_j) \\ &= \frac{1}{N} \text{Tr}((M_t - zI)^{-1} g(A)) \end{aligned} \quad (5.1)$$

where g is a real valued bounded function on \mathbb{R} . By convention, $g(A)$ is the diagonal matrix $\text{Diag}(g(a_1), g(a_2), \dots, g(a_N))$.

The interesting feature of $\Theta_N^g(z)$ is that, by the Stieltjes inversion formula, we have

$$\Phi_N(\lambda, \alpha) = \lim_{\eta \rightarrow 0^+} \frac{1}{\pi} \int_{-\infty}^{\lambda} \text{Im} [\Theta_N^g(\xi + i\eta)] d\xi$$

for the particular choice $g(x) = \mathbf{1}_{\{x \leq \alpha\}}$.

Thus, the problem is reduced to the study of the convergence of $\Theta_N^g(z)$ when $N \rightarrow \infty$. It is plain to deduce Theorem 3.2 from the following lemma.

Lemma 5.1. *Let g be a real valued bounded function on \mathbb{R} . Then, as $N \rightarrow +\infty$, we have the following convergence*

$$\Theta_N^g(z) \longrightarrow \Theta^g(z) = \int_{\mathbb{R}} \frac{g(a(x))}{a(x) - z - tG_{\mu_t}(z)} dx$$

where $G_{\mu_t}(z)$ is the Stieltjes transform of the limiting spectral distribution μ_t of the matrix M_t .

Proof of Lemma 5.1.

Using equation (5.1) and the definition of the matrix $R_t^N(z) = (M_t - zI)^{-1}$, it is straightforward to check that

$$\Theta_N^g(z) = \frac{1}{N} \sum_{i=1}^N g(a(\frac{i}{N})) \mathbb{E}[R_t^N(z)_{ii}].$$

Now, using Theorem 3.1 of Shlyakhtenko (see [14], and also [12, 19]), we know that the complex-valued measure σ_t^N defined in (3.5) converges weakly to $G_t(z, x) dx$. Therefore, as $N \rightarrow \infty$,

$$\Theta_N^g(z) \longrightarrow \int_{\mathbb{R}} g(a(x)) G_t(z, x) dx = \int_{\mathbb{R}} \frac{g(a(x))}{a(x) - z - tG_{\mu_t}(z)} dx,$$

using the fixed point equation (3.4) satisfied by $G_t(z, x)$. The lemma is proved. \square

A A reminder of perturbation theory

We can rewrite (2.1) as

$$M_t = A + \sqrt{t} H_1$$

where H_1 is a GOE matrix whose upper diagonal entries are independent Gaussian variables of variance $2/N$ on the diagonal and variance $1/N$ off the diagonal.

To second (respectively first) order in \sqrt{t} , we expect the following perturbative equations for the eigenvalue λ_i^t and the associated eigenvector $|\psi_i^t\rangle$ of the matrix M_t

$$\lambda_i^t = a_i + \sqrt{t} \alpha_i + t \beta_i + o(t), \quad (\text{A.1})$$

$$|\psi_i^t\rangle = (1 - \gamma_i t) |\phi_i\rangle + \sqrt{t} \sum_{j \neq i} \gamma_j |\phi_j\rangle + |\varepsilon_t\rangle. \quad (\text{A.2})$$

where $|\varepsilon_t\rangle$ is a vector such that $\langle \phi_i | \varepsilon_t \rangle = o(t)$ and $\langle \phi_j | \varepsilon_t \rangle = o(\sqrt{t})$ for $j \neq i$.

The aim here is to compute the terms α_i, β_i and γ_j for $j \neq i$. We start writing the eigenvalue equation for the matrix M_t

$$M_t |\psi_i^t\rangle = \lambda_i^t |\psi_i^t\rangle$$

which can be rewritten plugging equations (A.1) and (A.2) as

$$\begin{aligned} \sqrt{t} \sum_{j \neq i} \gamma_j a_j |\phi_j\rangle + \sqrt{t} H_1 |\phi_i\rangle + t \sum_{j \neq i} \gamma_j H_1 |\phi_j\rangle + A |\varepsilon_t\rangle &= \sqrt{t} \alpha_i |\phi_i\rangle + \sqrt{t} a_i \sum_{j \neq i} \gamma_j |\phi_j\rangle \\ &+ t \alpha_i \sum_{j \neq i} \gamma_j |\phi_j\rangle + t \beta_i |\phi_i\rangle + a_i |\varepsilon_t\rangle + |\tilde{\varepsilon}_t\rangle, \end{aligned}$$

where $|\tilde{\varepsilon}_t\rangle$ is a vector such that $\langle \phi_i | \tilde{\varepsilon}_t \rangle = o(t^{3/2})$ and $\langle \phi_j | \tilde{\varepsilon}_t \rangle = o(t)$ for $j \neq i$.

Projecting this equation on $|\phi_i\rangle^1$ and identifying the leading term in \sqrt{t} on both sides, we find

$$\alpha_i = \langle \phi_i | H_1 | \phi_i \rangle.$$

Then, projecting on $|\phi_j\rangle$ $j \neq i$ and identifying the leading term in \sqrt{t} on both sides, we find

$$\gamma_j = \frac{\langle \phi_j | H_1 | \phi_i \rangle}{a_i - a_j}.$$

Projecting again on $|\phi_i\rangle$ but identifying the term in t , we obtain

$$\beta_i = \sum_{j \neq i} \gamma_j \langle \phi_i | H_1 | \phi_j \rangle = \sum_{j \neq i} \frac{\langle \phi_j | H_1 | \phi_i \rangle^2}{a_i - a_j}.$$

The coefficient γ_i is computed with the additional normalization constraint $\langle \phi_i^t | \phi_i^t \rangle = 1$ to first order in t .

This gives the perturbation equations for the eigenvalue λ_i^t and the associated eigenvector $|\psi_i^t\rangle$. Note that this perturbative computation is valid only if the terms $t\beta_i, t\gamma_i$ and $\sqrt{t}\gamma_j$ remain very small compared to 1, corresponding to small values of t compared to the level spacings $a_i - a_j$ (which are of order $1/N$ in our setting above).

¹Note that the vectors $|\phi_k\rangle$ for $k = 1, \dots, N$ form an orthonormal family.

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