UNIVERSALITY CLASSES FOR EXTREME VALUE STATISTICS

Jean-Philippe Bouchaud[1] and Marc Mézard[2]

[1] *Service de Physique de l'État Condensé, Centre d'études de Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France*
[2] *Laboratoire de Physique Théorique de l'Ecole Normale Supérieure *, 24 rue Lhomond, 75231 Paris Cedex 05, France*

Abstract

The low temperature physics of disordered systems is governed by the statistics of extremely low energy states. It is thus rather important to discuss the possible universality classes for extreme value statistics. We compare the usual probabilistic classification to the results of the replica approach. We show in detail that one class of independent variables corresponds exactly to the so-called *one step replica symmetry breaking* solution in the replica language. This universality class holds if the correlations are sufficiently weak. We discuss the relation between the statistics of extremes and the problem of Burgers turbulence in decay.

LPTENS preprint 97/XX

Electronic addresses : bouchaud@amoco.saclay.cea.fr, mezard@physique.ens.fr

1 Introduction

The replica method is one of the very few general analytical methods to investigate disordered systems [1]. Although the physical meaning of Parisi's 'replica symmetry breaking' (RSB) scheme needed to obtain the correct low temperature solution of various random models has already been discussed on several occasions [1], its precise relation with the so-called extreme value statistics [2, 3] (and therefore its scope and limitations) was not previously clearly established. That such a relation should exist is however intuitively obvious: at low temperatures, a disordered system will preferentially occupy its low energy states, which are random variables because of the disordered nature of the problem. The statistics of the free-energy (or of other observables, such as energy *barriers* [4]) will thus reflect the statistics of these low energy (extreme) states. It is well known in probability theory that extreme value statistics can be classified into different universality classes [2, 3]. Conversely, the RSB scheme has shown the existence of at least two broad classes of systems, those with a first order, 'one step' RSB and those with continuous RSB.

It is easy to identify the 'one step RSB' class with one particular universality class of extreme value statistics, i.e. the Gumbel class, which concerns the minimum of continuous variables which are unbounded but have a distribution decaying faster than any power at $-\infty$. The simplest representative

*Unité propre du CNRS, associée à l'Ecole Normale Supérieure et à l'Université de Paris Sud of this class is the Random Energy Model (REM) [6], where the energy states are independent (but not necessarily Gaussian distributed). An interesting point is that the REM can be given a spatial structure, for which the replica theory still provides the *exact* solution. This spatial REM is in turn connected, in one dimension, to the problem of decaying Burgers' turbulence [8] in which an infinitely compressible fluid evolves from random initial conditions. Exact results for the velocity correlations at large times in Burgers' turbulence have been obtained long ago by Kida [9]. We shall show that these results coincide with those of the replica method, the underlying reason being that the late stage of turbulence decay is governed by the extreme values for the integral of the initial velocity field.

It is less easy to identify the other universality classes of extreme value statistics. There should be at least two types of generalizations. One type still concerns *independent* random variables but with either power law decay of the distribution (in which case there is a priori no replica formalism), or bounded random variables (the Weibull distribution of extremes), which does not seem to correspond to any known RSB scheme. The other type concerns *correlated* variables, for which the only results known to us are actually derived in the framework of replicas: those are cases of full RSB, which describe random variables with a certain (hierarchical) type of correlations.

These universality classes are the counterpart for extremes of random variables to the usual universality classes studied in the framework of sums of random variables. Taking the well known example of random walks or polymers, the usual random walk, or ideal polymer, is described asymptotically by the Gaussian central limit theorem, while the addition of independent variables with infinite variance leads to new universality classes (Lévy sums) [10]. The introduction of long-range correlations like self avoidance also leads to totally new universality classes [10]. We wish here to take a first step in an analogous categorization for extreme values, which appear naturally in disordered systems at low temperatures.

2 Extreme value statistics

2.1 Scaling regime

We start by recalling standard results of extreme values statistics, in order to set the stage for the following discussions. Consider M independent, identically distributed random variables E_i, $i = 1, ..., M$ ('energies'), such that the probability distribution decays for $E_i \to -\infty$ faster than any power-law:

$$P(E) \sim \frac{A}{|E|^\alpha} \exp[-B|E|^\delta] \qquad B, \delta > 0 \; ; \quad E \to -\infty \tag{1}$$

We are interested in the statistics of the lowest energy state $E^* = \min\{E_1, ...E_M\}$ for large M. Defining $\mathcal{P}_<(E)$ as the repartition function of E:

$$\mathcal{P}_<(E) = \int_{-\infty}^{E} dE' P(E') \tag{2}$$

one can express the distribution P_M of E^* as:

$$P_M(E^*) = MP(E^*)[1 - \mathcal{P}_<(E^*)]^{M-1} = -\frac{d}{dE^*}[\mathcal{P}_>(E^*)]^M \tag{3}$$

For large M, the minimum E^* will be negative and large, so that:

$$[1 - \mathcal{P}_<(E^*)]^M \simeq \exp[-M\mathcal{P}_<(E^*)] \tag{4}$$

The repartition function of E^* thus becomes very small when E^* is smaller than the characteristic value of the energy E_c defined by $M\mathcal{P}_<(E_c) = 1$. To logarithmic accuracy, this gives in the case of the distribution (1):

$$E_c \simeq -\left(\frac{\log M}{B}\right)^{1/\delta} \tag{5}$$

Defining now $E^* = E_c + \epsilon$, with $\epsilon \ll |E_c|$, one has, to first order:

$$[1 - \mathcal{P}_<(E^*)]^M \simeq \exp[-\exp(B\delta|E_c|^{\delta-1}\epsilon)] \tag{6}$$

Finally, introducing the rescaled variable $u = B\delta|E_c|^{\delta-1}\ \epsilon$, one finds that the rescaled variable obeys, for large M, a universal 'Gumbel' distribution [2, 3] P^*, independent of the coefficients A, B and exponents α, δ:

$$P^*(u) = \exp(u - \exp u) \tag{7}$$

A very important property, which we shall emphasize later on, is that $P^*(u)$ vanishes exponentially for $u \to -\infty$ (and much faster still for $u \to +\infty$). The maximum of $P^*(u)$ occurs at $u = 0$, meaning that E_c is actually the most probable value for the extreme energy. Finally, as in any 'central' limit theorem, this behaviour is only valid in the region where the deviation ϵ from E_c is of the order of $E_c^{1-\delta}/B$, which goes to zero with M if $\delta > 1$ and diverges otherwise. The *relative* fluctuations ϵ/E_c, however, are always of order $1/\log M$.

2.2 The large M limit and the Random Energy Model

Let us now consider the following partition function:

$$\mathcal{Z} = \sum_{i=1}^{M} z_i \qquad z_i = \exp[-\frac{E_i}{T}] \tag{8}$$

where the E_i are distributed as in (1). This is a slight generalization of Derrida's original REM, initially introduced with a purely Gaussian distribution ($\delta = 2$). Obviously, the independent variables z_i, are large when E_i is large and negative. In the scaling region defined above, due to the exponential tail of (7), the distribution of z decays for large z as a power-law:

$$P(z) \propto z^{-1-\mu} \qquad ; \qquad z \to \infty \tag{9}$$

where E_c is the most probable ground state energy of the system, given by Eq. (5), and

$$\mu = TB\delta|E_c|^{\delta-1}\ . \tag{10}$$

The partition sum $\mathcal{Z}$ behaves very differently in the region $\mu < 1$, where the average value of z diverges and thus only a small number of terms (those of order $M^{1/\mu}$) contribute to $\mathcal{Z}$, and in the region $\mu > 1$, where all the M terms give a (small) contribution to $\mathcal{Z}$. This means that for

$$T_c = \frac{1}{B\delta E_c^{\delta-1}} \tag{11}$$

for which $\mu = 1$, the probability measure concentrates onto a finite number of states, corresponding to the glass transition in these models. In the random energy model, M is the number of states $M = 2^N$. In order to have an extensive ground state energy ($E_c \propto N$) and T_c finite in the large N limit, one should choose (see Eqs. (5,11)) $B = N^{1-\delta}$. For $\delta = 2$, this indeed coincides with the usual scaling of the energies in the REM.

Let us now study the statistics of the weights $p_i \equiv z_i/\mathcal{Z}$ in the glassy region $T < T_c$. Since:

$$w_i = \frac{z_i}{z_i + \mathcal{Z}'} \tag{12}$$

where $\mathcal{Z}' = \sum_{k(\neq i)} z_k$ is independent of z_i (and of order $M^{1/\mu}$), one readily finds that [1]:

$$P(w) = \frac{\mathcal{Z}'}{(1-w)^2} P\left(z = \frac{\mathcal{Z}' w}{1-w}\right) \tag{13}$$

For w_i to be non zero in the large M limit, z_i has to be large. In that region one can use the asymptotic form (9) for $P(z)$, giving:

$$P(w) = \frac{C}{M}(1-w)^{\mu-1} w^{-1-\mu} \qquad w \gg M^{-1/\mu} \tag{14}$$

where C is a constant fixed by the condition $M \int_0^1 dw w P(w) \equiv 1$. From this probability distribution of each weight, one can deduce the moments $Y_k \equiv \overline{\sum_i w_i^k}$, which characterize to what extent the measure concentrates onto a few states: if all weights are of the same order of magnitude, then $Y_k \sim M^{1-k} \to 0$ for $k > 1$; while if only a finite number of weights contribute, the moments Y_k remain finite when $M \to \infty$. In the present case, one finds, for $\mu < 1$,

$$Y_k = M \int_0^1 dw w^k P(w) = \frac{\Gamma[k-\mu]}{\Gamma[k]\Gamma[1-\mu]} \qquad (k > \mu) \tag{15}$$

(see also [11]). Since $\mu = T/T_c$, one finds that Y_2 goes linearly to zero for $T \to T_c$, and that $Y_k = 1 - (\Gamma'[k] - \Gamma'[1])/\Gamma[k])T/T_c$ for $T \to 0$.

Finally, the average energy per degree of freedom of the system is constant throughout the low temperature phase ($T < T_c$) and given by

$$\overline{E}/N = E_c/N + < u > B\delta E_c^{\delta-1}/N \sim -(\log 2)^{1/\delta} + O(1/N) \ , \tag{16}$$

where the average $< u >$ is taken over the Gumbel distribution, giving: $< u >= \Gamma'[1]$.

3 The replica approach

3.1 The REM

We shall now show how all these results can be recovered using the replica method. We suppose that $\delta > 1$ (the case $\delta < 1$ will be discussed below) and introduce the characteristic function $g(\lambda)$ through:

$$\int_{-\infty}^{\infty} dE P(E) \exp[-\lambda E] \equiv \exp[g(\lambda)] \tag{17}$$

Since $B = N^{1-\delta}$, this integral can be computed at large N with a saddle-point method, which gives:

$$g(\lambda) = (\delta - 1) N \left(\frac{\lambda}{\delta}\right)^{\frac{\delta}{\delta-1}} \tag{18}$$

[1] We denote as $P(.)$ the probability density of the variable appearing in the parenthesis; hopefully there is not ambiguity in the following.

In the replica method we need to compute the moments of the $\mathcal{Z}$ distribution:

$$\overline{\mathcal{Z}^n} = \overline{\sum_{i_1,i_2,...i_n} z_{i_1} z_{i_2} ... z_{i_n}} \equiv \overline{\sum_{i_1,i_2,...i_n} \exp\left[-\frac{1}{T}\sum_i E_i \sum_{a=1}^n \delta_{i,i_a}\right]} \tag{19}$$

The averaging over the E_i gives:

$$\overline{\mathcal{Z}^n} = \sum_{i_1,i_2,...i_n} \exp[\sum_i g(\frac{1}{T}\sum_{a=1}^n \delta_{i,i_a})] \tag{20}$$

The point now is to understand which configurations of $\{i_1, i_2, ...i_n\}$ will dominate the above sum when $N \to \infty$ (and $n \to 0$). The simplest Ansatz, corresponding to the largest phase space volume, assumes that all i_a are different, leading to:

$$\overline{\mathcal{Z}^n} = M(M-1)...(M-n+1)\exp[ng(\frac{1}{T})] \simeq \exp[n(\log M + g(\frac{1}{T}))] \tag{21}$$

Taking $n \to 0$, one thus finds that the free energy per degree of freedom $f = -\frac{T}{N}\overline{\log \mathcal{Z}}$ takes the value:

$$f = f_0 \equiv -T\log 2 - (\delta - 1)\delta^{-\frac{\delta}{\delta-1}} T^{-\frac{1}{\delta-1}} \tag{22}$$

The entropy $s_0 = -df_0/dT$ is therefore equal to:

$$s_0 = \log(2) - (\delta T)^{-\frac{\delta}{\delta-1}} \tag{23}$$

and becomes negative below a critical temperature

$$T_c = \frac{1}{\delta}\log(2)^{\frac{1-\delta}{\delta}} \tag{24}$$

So this solution, called 'replica symmetric' (since all replicas i_a play a symmetric role), has to be modified in the low temperature phase. The correct configurations which dominate the sum (20) at $T < T_c$ are called 'one step replica symmetry breaking' and are such that the n replica indices $\{i_1, i_2, ...i_n\}$ are grouped into n/m groups of m equal indices, which can be written after a proper relabelling:

$$i_1 = i_2 = ... = i_m = k_1 \tag{25}$$

$$i_{m+1} = i_{m+2} = ... = i_{2m} = k_2 \tag{26}$$

$$... \tag{27}$$

$$i_{n-m+1} = ... = i_n = k_{n/m} \tag{28}$$

and now the indices $k_1, ..., k_{n/m}$ are all different one from the other. These configurations contribute to $\overline{\mathcal{Z}^n}$ as:

$$\overline{\mathcal{Z}^n} = M(M-1)...(M-n/m+1)\exp[\frac{n}{m}g(\frac{m}{T})]\frac{n!}{m!^{n/m}} \tag{29}$$

from which one immediately deduces:

$$f(T) = f_0(T/m) \tag{30}$$

where f_0 is defined in Eq. (22). The extremum of this free energy with respect to m is obtained when

$$\frac{\partial f}{\partial m} = 0 = s_0(T/m) \tag{31}$$

which gives

$$m = \frac{T}{T_c} = \mu \tag{32}$$

Note that this relation is independent of δ. Therefore this one step RSB solution predicts that the system freezes at the critical temperature T_c which is the temperature where the entropy s_0 vanishes. The energy density is constant throughout the low temperature phase, and equals:

$$e = f_0(T_c) = -(\log\ 2)^{1/\delta} \tag{33}$$

in agreement with the direct computation (16). Since the free-energy is constant, the entropy of the whole low temperature phase is zero [6].

It turns out that also the finer details, like the distribution of the weights of the configurations which dominate the low temperature measure, can be computed by this replica approach [12]. By definition, the moments Y_k are equal to:

$$\begin{aligned} Y_k &= \overline{\sum_i \frac{z_i^k}{\mathcal{Z}^k}} = \lim_{n\to 0} \overline{\sum_i z_i^k \mathcal{Z}^{n-k}} & (34)\\ &= \lim_{n\to 0} \frac{1}{n(n-1)...(n-k+1)} \sum_{a_1,..,a_k}{}' \sum_{i_1,...,i_n} \overline{z_{i_1}...z_{i_n}} \prod_{j=1}^{k} \delta_{i_{a_1},i_{a_j}} & (35) \end{aligned}$$

where the sum primed over the a's runs from 1 to n, with all a's different. Owing to the structure of the RSB, this means that one simply has to pick the $k \leq m$ replica indices $a_1, ..., a_k$ in the same 'group', for which there are $(m-1)...(m-k+1)$ possibilities once a_1 has been chosen. Hence:

$$Y_k = \lim_{n\to 0} \frac{n(m-1)...(m-k+1)}{n(n-1)...(n-k+1)} \overline{\mathcal{Z}^n} = \frac{\Gamma[k-\mu]}{\Gamma[k]\Gamma[1-\mu]} \tag{36}$$

in agreement with the direct computation (15).

3.2 The REM with $\delta < 1$: a first order transition

The above method fails when $\delta \leq 1$, which actually corresponds to a different universality class from the point of view of critical phenomena, while the nature of the low temperature phase leaves it in the same class as the systems with $\delta > 1$, in agreement with the extreme value classification which does not distinguish between $\delta > 1$ or $\delta < 1$. In order to study the transition, we use Derrida's original 'microcanonical' method. Using the normalisation $B = N^{1-\delta}$, the partition function is equal to:

$$\mathcal{Z} = \int_{-e_c}^{0} de \exp N\varphi(e) \qquad \varphi(e) = \log 2 - |e|^\delta + \frac{|e|}{T} \tag{37}$$

where $e = E/N$, and e_c is the energy density beyond which there are no states (for $N \to \infty$), i.e.: $2^N \exp(-Ne_c^\delta) = 1$. As shown in Fig. (1), the integral is dominated either by $e = 0$ or by $e = -e_c$, depending on the temperature. When $T > T_c = (\log 2)^{1-\delta/\delta}$, the free energy is equal to $-NT \log 2$, while for $T < T_c$, the free energy is equal to a constant $-Ne_c = -N(\log 2)^{1/\delta}$. The transition at T_c is now a first order transition from the thermodynamic point of view, with a jump in the entropy. This is in contrast with the usual case $\delta > 1$ where the transition is thermodynamically of second order [2].

[2]Although there is a jump in the Edwards-Anderson order parameter [6]

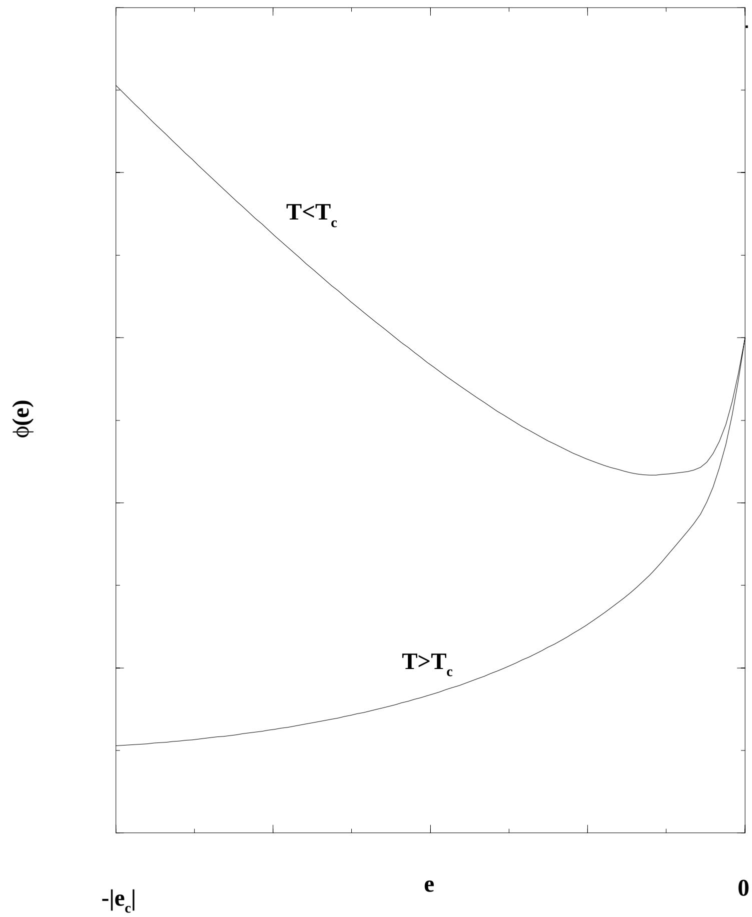

Figure 1: $\varphi(e)$, as defined in Eq. (37), as a function of e for different temperatures. The saddle point is thus at $e = 0$ for $T > T_c$ and at $e = -|e_c|$ for $T < T_c$. There is no states (in the limit $N \to \infty$ beyond $-|e_c|$.

In the low temperature phase, only the neighbourhood of $E_c = -N|e_c|$ is of importance, and we get back to the universal Gumbel distribution since the density of states is still locally exponential:

$$P(E = E_c + \epsilon) \propto \exp \frac{\mu\epsilon}{T} \qquad \mu \equiv \frac{T}{T_c}\delta \tag{38}$$

The value of μ again determines the statistics of the weights, as above. Note that however that for $\delta < 1$, the value of the parameter μ (corresponding to the RSB parameter) is smaller than 1 at the transition $T = T_c^-$. Hence Y_2 is discontinuous at $T = T_c$, in contrast to the case $\delta > 1$.

3.3 Physical interpretation of the replica solution

The reason why replica symmetry must be broken in order to get sensible results in this problem is rather clear. Since the distribution of the Boltzmann weights z_i is a power law with an exponent $\mu < 1$ in the low temperature phase, all integer moments of $\mathcal{Z}$ (and thus of z) are formally divergent, and are thus dominated by a cut-off for large z which has nothing to do with the value of the 'typical' z's, and hence of the free-energy. Calculation based on a simple analytic continuation of the results obtained for $n > 1$ are thus bound to fail. The replica method with one step RSB manages to compute $\langle z^m \rangle$ with $m = \mu$, which precisely picks up the contribution of the typical region of z.(Smaller values of m would be mostly sensitive to very small z, while larger m's probe atypically large values of z.) The algebra corresponding to one step RSB exactly reproduces the extreme value statistics in the case of fastly decaying distributions. In this respect, RSB does not mean more than a 'localisation' of weights onto a small subset of all configurations, in the sense that major contribution to the partition function comes from a finite number of configurations (i.e. all $Y_k > 0$) [12]. Actually, the quantities Y_k were also introduced in the context of electron localisation in disordered potentials, and called 'participation ratios' [13].

4 A d-dimensional Random Energy Model

In this section we want to study a generalized version of the REM, where the energy levels are embedded in a euclidean space. Besides its intrinsic interest as a model for a particle in a disordered environment, this problem turns out to be also directly relevant to the study of declining Burgers turbulence, as we shall discuss in detail in the next section.

The model is defined as follows. To each point x of a (discretized) d-dimensional space, one assigns a potential energy $E(\vec{x})$ which is a random number picked up independently on each point, from a distribution $P(E)$ the tail of which is given by (1). The total energy on this point is the sum of a deterministic part, which we take for instance equal to $\kappa x^2/2$, and this random contribution $E(\vec{x})$. This defines a certain energy landscape, to which we associate a partition function $\mathcal{Z}$ as:

$$\mathcal{Z} = \int d^d x \exp\left(-\frac{V(\vec{x})}{T}\right) \quad , \quad V(\vec{x}) \equiv \frac{\kappa x^2}{2} + E(\vec{x}) \tag{39}$$

Here we adopt a continuum notation but an ultraviolet cutoff (lattice spacing) is implicitely assumed when necessary. The role of the deterministic part proportional to κ is twofold. First of all it allows one to define a topology in the space of the points $\vec{x}$ (The limit $\kappa = 0$coincides with the REM, the fact that the points sit in a d-dimensional space being irrelevant). For this purpose the deterministic part could be rather arbitrary, and indeed one can solve the problem with a more general deterministic energy. As we shall need a quadratic term later on, and in order to keep the presentation simple, we restrict to this particular case. Second, the presence of this confining term allows one to deal with this model without the need of introducing a finite box.

This model with $d = 1$ was in fact introduced and studied long ago as a toy model of an interface in a random medium [14]: one possible interpretation is that x is the coordinate of a particular point on the interface, which feels a random pinning potential $E(x)$, while the quadratic potential is a mean-field description of the elasticity due to the rest of the interface. Another interpretation (in the context of Bloch walls) is to neglect the deformation of the interface, which is only described by its center of mass coordinate x. The quadratic potential is then induced by the demagnetizing fields due to the surrounding Bloch walls.

We want to compute the low temperature properties of this system in the limit when $\kappa \to 0$. For instance one would like to know the typical displacement of the ground state, measured through $\overline{< x^2 >}$, or the average ground state energy, etc. In the special case where the energy is Gaussian distributed, this problem has already been studied by scaling arguments [14], or with a Gaussian replica variational method [15]. We shall provide hereafter the exact solution, first using a direct extreme value statistics approach and then with the replica method.

4.1 Extreme value approach

For simplicity, we restrict to the case $d = 1$; the extension to higher dimensions is however immediate. For temperatures going to zero, we want to find the minimum of all the energies $\frac{\kappa}{2}x^2 + E(x)$ when x scans a one dimensional lattice. The joint probability that this minimum is achieved on a point x^* and takes a value $V(x) = \frac{\kappa}{2}x^{*2} + E$ is given by:

$$P(x^*, E) = P(E) \prod_{x' \neq x^*} \left(1 - P_< \left(E + \frac{\kappa}{2}x^{*2} - \frac{\kappa}{2}x'^2\right)\right) \tag{40}$$

For $\kappa \to 0$ we can safely take a continuum limit and we get:

$$P(x^*, E) = \frac{P(E)}{1 - P_<(E)} \exp\left(\int dx' \log\left[1 - P_< \left(E + \frac{\kappa}{2}x^{*2} - \frac{\kappa}{2}x'^2\right]\right)\right) \tag{41}$$

Integrating over E we get the probability that the minimum is achieved on point x^*. For small κ, the minimum E is expected to be negative and large, and hence only the region where $P_<$ is small will be of importance. Rescaling x^* as $x^* = \hat{x}^*/\sqrt{\kappa}$, we obtain

$$P(\hat{x}^*) \simeq \int dE P(E) \exp\left(-\int \frac{dz}{\sqrt{\kappa}} P_< \left(E + \frac{\hat{x}^{*2} - z^2}{2}\right)\right) \tag{42}$$

For small κ it is thus clear that the relevant energy region is the one around the value E_c such that $P_<(E_c) = \sqrt{\kappa}$, or:

$$E_c = -\left(\frac{\log(1/\sqrt{\kappa})}{B}\right)^{1/\delta} \tag{43}$$

(Notice that the role of the number M of energy levels in the first section is played here by the length scale $1/\sqrt{\kappa}$, which is natural.) Expanding the energy around E_c as $E = E_c - \hat{x}^{*2}/2 + \epsilon$, we get:

$$P_< \left(E + \frac{\hat{x}^{*2} - z^2}{2}\right) \sim \sqrt{\kappa} \exp\left(\delta B |E_c|^{\delta-1}(\epsilon - \frac{z^2}{2})\right) \tag{44}$$

The integral over z in (43) is a thus a Gaussian integral. We finally get, after a simple integration over ϵ:

$$P(\hat{x}^*) \propto \exp\left(-\delta B |E_c|^{\delta-1} \frac{\hat{x}^{*2}}{2}\right) \tag{45}$$

Therefore we have shown that the typical distance to the origin of the point x^* corresponding to a minimum energy is

$$\xi = \left(\kappa\delta B|E_c|^{\delta-1}\right)^{-1/2} = \frac{1}{\sqrt{\kappa\delta}}\left(\log\left(\frac{1}{\sqrt{\kappa}}\right)\right)^{\frac{1-\delta}{2\delta}} B^{-\frac{1}{2\delta}} \tag{46}$$

and more precisely the distribution of x^*/ξ is a Gaussian of unit variance [3].

We can also compute the probability distribution of the ground state energy $\mathcal{V}^*$ as

$$P(\mathcal{V}^*) = \int dx \int dE\ \delta\left(\mathcal{V}^* - \frac{\kappa}{2}x^2 - E\right) P(x,E) \tag{47}$$

where $P(x,E)$ is given in (41). The result is the following: introducing the rescaled energy u as:

$$\mathcal{V}^* = E_c + \frac{1}{2B\delta|E_c|^{\delta-1}}\log\left[\frac{B\delta|E_c|^{\delta-1}}{2\pi}\right] + \frac{u}{B\delta|E_c|^{\delta-1}} \tag{48}$$

one finds that u is distributed according to the universal Gumbel distribution, Eq. (7). In particular, the extremely deep states are exponentially distributed, as $\exp[\mu\mathcal{V}^*/T]$, with $\mu = TB\delta|E_c|^{\delta-1}$.

4.2 Replica approach

Interestingly, the replica approach with a one step RSB also leads to the *exact* result. Introducing again the generating function $g(\lambda)$ of $P(E)$, we have, for large λ:

$$g(\lambda) = \frac{\delta-1}{B}^{-1/\delta-1}\left(\frac{\lambda}{\delta}\right)^{\delta/\delta-1} \tag{49}$$

The average of $\mathcal{Z}^n$ can thus be expressed as:

$$\overline{\mathcal{Z}^n} = \sum_{x_1,..x_n}\exp\left[-\frac{\kappa}{2T}\sum_{a=1}^{n} x_a^2 + \sum_x g\Big(\frac{1}{T}\sum_{a=1}^{n}\delta_{x,x_a}\Big)\right] \tag{50}$$

Let us make the ansatz that at low temperature, the saddle point of this expression is such that:

$$x_1 = x_2 = ... = x_m = y_1 \tag{51}$$
$$x_{m+1} = x_{m+2} = ... = x_{2m} = y_2 \tag{52}$$
$$... \tag{53}$$
$$x_{n-m+1} = ... = x_n = y_{n/m} \tag{54}$$

and perform the Gaussian integration over the y_i. We finally obtain:

$$\overline{\mathcal{Z}^n} = \exp\frac{n}{m}\left[\frac{1}{2}\log\frac{2\pi T}{\kappa m} + g\Big(\frac{m}{T}\Big)\right] \equiv \exp -\frac{n}{T}f(\rho) \tag{55}$$

with $\rho = m/T$. Looking for the extremum of f as a function of ρ we find, in the limit $\kappa \to 0$ (and with $\delta > 1$),

$$\rho^* = \delta B^{1/\delta}\Big(\log\frac{1}{\sqrt{\kappa}}\Big)^{(\delta-1)/\delta} \equiv B\delta|E_c|^{\delta-1} \tag{56}$$

where E_c is given by Eq. (43).

[3] Notice that for small κ we have $\kappa\xi^2 << |E_c|$ which justifies our expansion around E_c in the derivation of $P(\hat{x}^*)$

It is easy to check that the free energy $f(\rho^*)$ precisely reproduces the above result for the ground state energy obtained directly, Eq. (48). As explained above, the calculation of the quantities Y_k within the replica method indicates that the low energy states are exponentially distributed with a parameter given by $\rho^* = m/T$. Hence, comparing (48) and (56), we see that the replica method indeed predicts the correct statistics of deep states. The replica method also allows one to calculate

$$\overline{P(x)} = \sum_{x_2,..x_n} \exp\left[-\frac{\kappa}{2T}\sum_{a=1}^{n} x_a^2 + \sum_x g(\frac{1}{T}\sum_{a=1}^{n}\delta_{x,x_a})\right]\Bigg|_{x_1=x} \tag{57}$$

Within the above one step solution, this immediately leads to the following Gaussian result:

$$\overline{P(x)} = \sqrt{\frac{\kappa\rho^*}{2\pi}} \exp -\frac{\kappa\rho^* x^2}{2} \tag{58}$$

which is identical to Eq. (45).

The replica method also allows one to discuss the non zero temperature regime, which is much harder to study directly. As shown above, there is a phase transition towards a 'delocalised' phase where $Y_k \equiv 0$ when $\mu = \rho^* T = 1$ [4]. However, for any non-zero temperature and for $\delta > 1$, the system eventually reaches $\mu = 1$ for small enough κ. This can be interpreted as follows: as $\kappa \to 0$, the number of accessible states diverges. But since the difference between the ground state and the first excited state decreases as $|E_c|^{1-\delta}$ (when $\delta > 1$), it does become smaller than T for a sufficiently small κ, beyond which a large number of quasi-degenerate states contribute to the partition function, as in the high temperature phase. Only for $\delta = 1$ is there a true transition temperature, independent of κ (see [16] for a discussion of this point in a different context). For $\delta < 1$, one expects a first order phase transition (see above).

Finally, let us note that in the case where the confining potential is harmonic (i.e. equal to $\kappa x^2/2$), the Gaussian *variational* replica method developed in [7, 17] also gives the *exact* result for ρ^*.

4.3 Physical interpretation of the replica solution

Using the replica method, one can also compute higher moments of $P(x)$, such as $\overline{P(x)P(y)}$, etc. One can then show that the replica solution is identical to the following probabilistic construction for $P(x)$ for a given sample:

$$P(x) = \frac{1}{\mathcal{Z}} \sum_\alpha w_\alpha \delta_{x,x_\alpha} \tag{59}$$

where the w_α are random weights, chosen with a probability distribution given by Eq. (14), and the x_α are random variables, independent from the w's, and chosen according to a Gaussian of width κ^{-1}.

5 The random energy model and Burgers' turbulence

5.1 The Cole-Hopf transformation

It is well known that the solution of Burgers equation with a random initial velocity field can be expressed as a partition sum of the form Eq. (39). Let us restrict for simplicity to one dimension, although, again, generalisation to higher dimensions is possible. The Burgers equation in the absence of forcing reads:

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu\frac{\partial^2 v}{\partial x^2} \tag{60}$$

[4] Note however that there is a true phase transition only in the limit $\kappa \to 0$ or $d \to \infty$, i.e. when the number of degrees of freedom is infinite. Otherwise, the transition for $\mu = 1$ is really a crossover.

where ν is the viscosity. The initial velocity field $v(x, t = 0)$ will be chosen as $v(x, t = 0) = \frac{\partial E(x)}{\partial x}$. Writing $v = -2\nu \frac{\partial \log \mathcal{Z}}{\partial x}$, allows to transform the Burgers equation into the following linear diffusion equation (Cole-Hopf transform):

$$\frac{\partial \mathcal{Z}}{\partial t} = \nu \frac{\partial^2 \mathcal{Z}}{\partial x^2} \tag{61}$$

with initial condition $\mathcal{Z}(x, t = 0) = \exp[-E(x)/2\nu]$. The solution thus reads:

$$\mathcal{Z}(x_0, t) = \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{4\pi\nu t}} \exp\left(-\frac{1}{2\nu}[\frac{(x - x_0)^2}{2t} + E(x)]\right) \tag{62}$$

which is, up to a multiplicative factor, identical to the 'spatial' REM defined by Eq.(39) with the following identification:

$$T \to 2\nu \qquad \kappa \to \frac{1}{t} \tag{63}$$

Physically, the disordered problem associated to this REM is that of a point particle interacting with a (random) pinning potential $E(x)$, attached by a spring to point x_0, which is a simplified model for an extended elastic object in a random potential. This model was also recently considered in the context of solid friction [18].

Although the two problems, spatial REM on one hand and decaying Burgers turbulence on the other hand, are formally identical, they may differ by the type of questions one wants to address. For instance in turbulence one is interested in the correlations of velocities, which involves knowing the variations of the free energy of the REM (62) when x_0 varies. The case where $E(x)$ is random with short range correlations correspond to a short range correlated velocity field $v(x, t = 0)$ with a 'blue' spectrum (i.e. $|v(k, t = 0)|^2 \propto k^2$, where k is the Fourier variable) and the small viscosity (large Reynolds) limit corresponds to small temperature in the associated disordered problem.

5.2 Cusps and shocks

In the zero viscosity (or zero temperature) limit, the partition function (62) can be evaluated by a saddle point method. For a fixed x_0, one looks for the value of x^* such that $\kappa(x_0 - x^*)^2/2 + E(x^*)$ is minimum. The saddle point construction [8] is graphically explained in Fig. (2), for a simple profile $E(x)$. For a given x_0, one draws as a function of x the parabola $\mathcal{V} - \kappa(x_0 - x)^2/2$ and looks for the minimum value of $\mathcal{V}$, called $\mathcal{V}^*(x_0)$, such that this parabola intersects the curve $E(x)$; calling x^* the intersection point, the saddle point approximation gives:

$$\mathcal{Z}(x_0, t) \simeq \exp[-\frac{\mathcal{V}^*(x_0)}{2\nu}] \simeq \exp\left(-\frac{1}{2\nu}[\frac{\kappa}{2}(x_0 - x^*)^2 + E(x^*)]\right) \tag{64}$$

For large values of κ, the parabola is very sharp, and there is only one 'optimal' intersection point x^* for each value of x_0; to a first approximation, one thus has $\mathcal{Z}(x_0, t) \simeq \exp[-E(x_0)/T]$. On the other hand for very small κ, which corresponds to the large time limit of the decaying Burgers turbulence, the parabola $\mathcal{V} - \kappa(x_0 - x)^2/2$ is extremely flat and the intersection points will be determined by the extreme (negative) values of the potential $E(x)$. In this limit, the statistics of the effective potential $\mathcal{V}^*$ – and thus of the velocity field $v(x, t)$ – reflects the statistics of the extreme values of $E(x)$, and is thus, to a large degree, universal. Generically, the solution x^* depends very weakly on x_0 and the effective potential $\mathcal{V}^*(x_0)$ can thus approximatively be written as:

$$\mathcal{V}^*(x_0) \simeq \frac{\kappa}{2}(x_0 - x^*)^2 + E(x^*) \tag{65}$$

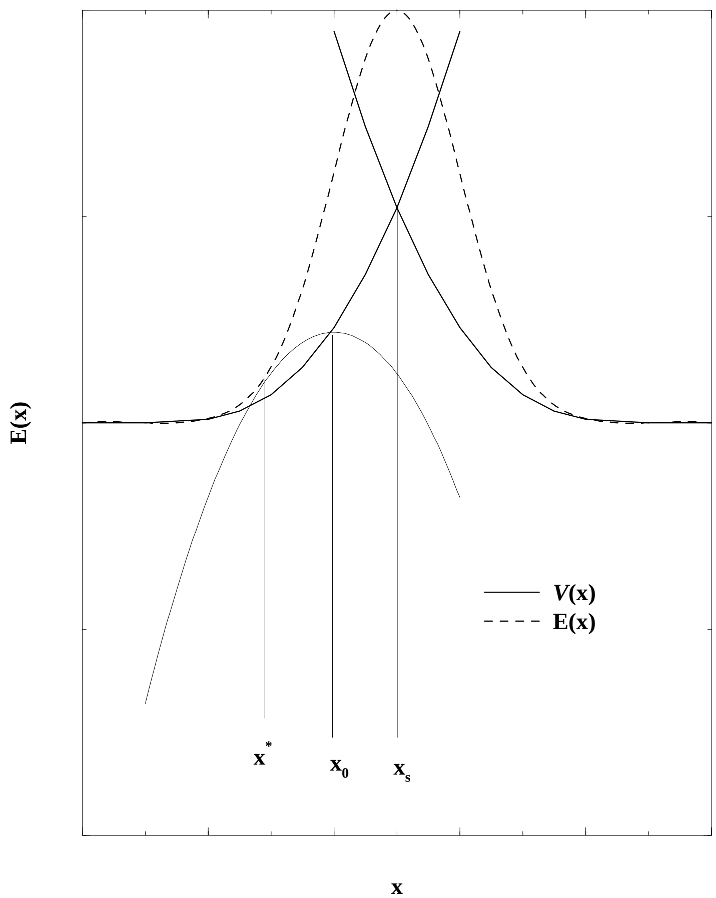

Figure 2: Graphical solution of the Burgers equation in the limit of small viscosity, in the neighbourhood of a cusp. The dashed line is the original potential $E(x)$, while the full line corresponds to the effective potential $\mathcal{V}^*(x)$. The curves actually continue beyond the cusp of $\mathcal{V}^*(x)$, where one metastable and one stable saddle point coexist.

with a *fixed* x^*, where $E(x^*)$ corresponds to a particularly 'deep' minimum x^* of the potential $E(x)$. This is the generic situation when one varies x_0 locally; it corresponds to a velocity field which is locally linear:

$$v(x_0) = \frac{d\mathcal{V}^*(x_0)}{dx_0} = \kappa(x_0 - x^*) \tag{66}$$

(Remember that by definition $\partial\mathcal{V}^*/\partial x^* = 0$). There exist however exceptional values x_s of x_0 such that the first intersection of the parabola and the curve $E(x)$ appears simultaneously at two points $x_1^* < x_2^*$: when x_0 varies from $x_s - \epsilon$ to $x_s + \epsilon$, the solution x^* jumps from x_1^* to x_2^*. This corresponds to a cusp in the minimum value $\mathcal{V}^*$ as a function of x_0 (see Fig. (2)). In the language of Burgers' turbulence, this is a shock since the velocity v (which is the derivative of $\mathcal{V}^*$) is discontinuous at $x_0 = x_s$.

5.3 Decay from an uncorrelated $E(x)$ configuration: Kida's analysis

Let us now focus onto the case where $E(x)$ is randomly distributed with a short range correlation, and the time t is large, corresponding to a very small κ. This limit was studied in detail by Kida in the context of Burgers' equation [9] (see also [19]). Let us denote by x_α the various values of the intersection points between the parabola and the curve $E(x)$ when one varies x_0. After a proper coarse graining one can totally forget about the correlations of $E(x)$, and thus the $\{x_\alpha\}$ are randomly (Poisson) distributed along the x-axis. If the distribution of E decays as $\exp -B|E|^\delta$, the extreme value statistics tells us that the distribution of $E_\alpha \equiv E(x_\alpha)$ is of the Gumbel type. The only delicate point is to understand what is the effective number of independent variables, M, appearing in this distribution. This number depends on κ and is determined self-consistently as follows: as x_0 departs from x_α, at some point (because of the quadratic growing term $\kappa(x_0 - x_\alpha)^2/2$) will a better saddle point x_β be prefered. Since the width of the Gumbel distribution is given by:

$$\frac{1}{\delta B^{1/\delta}} (\log M)^{\frac{1-\delta}{\delta}} \tag{67}$$

(see Eq. (6) above), this sets the order of magnitude of the difference between E_α and E_β, which must also be, by definition, of the order of $\kappa(x_\alpha - x_\beta)^2$. Furthermore, taking the correlation length of the potential $E(x)$ to be 1, the effective number of independent variables is given by:

$$M = |x_\beta - x_\alpha| \simeq \frac{1}{\sqrt{\kappa\delta B^{1/\delta}}} (\log M)^{\frac{1-\delta}{2\delta}} \tag{68}$$

or, to logarithmic accuracy, and using the correspondance $\kappa \to 1/t$,

$$M \propto \sqrt{t}\,(\log t)^{\frac{1-\delta}{2\delta}} \tag{69}$$

Note that by definition, M is also the typical distance beween two shocks $\ell(t)$, which is thus seen to grow as $t^{1/2}$ with logarithmic corrections (these corrections disappear for $\delta = 1$, where the initial potential already possesses the universal exponential tail). This is one of the important results of the original analysis of Kida. Furthermore, since the local slope of the velocity is $\kappa = 1/t$, the maximum velocity is of order:

$$v_{\max} = \frac{\ell(t)}{t} \simeq \frac{(\log t)^{\frac{1-\delta}{2\delta}}}{\sqrt{t}} \tag{70}$$

which corresponds to a time dependent Reynolds number :

$$Re = \frac{v_{\max}\ell}{\nu} \propto \frac{(\log t)^{\frac{1-\delta}{\delta}}}{\nu} \tag{71}$$

which goes to zero (albeit very slowly) when $t \to \infty$ for $\delta > 1$. This is similar to the above remark that for any small temperature, the system goes back into its high temperature phase when $\kappa \to 0$.

Using this construction, and the full distribution of the E_α, Kida was able to obtain directly the large time behaviour of the two point velocity correlation, $\overline{v(x)v(x+r)}$, which is a universal function once the lengths are expressed in terms of the mean distance between two shocks $\ell(t)$, and the velocities in units of $v_{\max}$ [9]. His result is recalled in the appendix. Let us show how one can obtain precisely the same results using the replica method, which in fact provides the ful probability distribution function of $v(x+r) - v(x)$.

5.4 The replica analysis

Let us first note that Eq. (65) can alternatively (in the limit $\kappa, T \to 0$) be written as an infinite sum:

$$\mathcal{Z}(x_0) = \exp[-\mathcal{V}^*(x_0)/T] = \sum_\alpha w_\alpha \exp[-\frac{\kappa(x_0 - x_\alpha)^2}{2T}], \tag{72}$$

where x_α are Poisson distributed with an arbitrary (see below) linear density σ. The w_α are independent random variables again chosen according to the distribution (14), with μ given by (see Eq. (10)):

$$\mu = T\delta B^{1/\delta}(\log M)^{-\frac{1-\delta}{\delta}} \tag{73}$$

That Eq. (72) precisely reproduces Kida's construction comes from the fact that, as $T \to 0$, the distribution of weights becomes so broad that the sum determining $\mathcal{Z}(x_0)$ becomes entirely dominated by a single term, which is the one which maximizes $w_\alpha \exp[-\kappa(x_0 - x_\alpha)^2/2]$. Again, the corresponding x_α switches discontinuously as a function of x_0, when another value x_β suddenly takes over. This construction is independent of the density σ, provided that $\sigma M \gg 1$ (i.e., in the long time limit).

The crucial point now is that the explicit construction (72) actually gives results which are *identical* to those obtained using a replica representation:

$$\overline{\mathcal{Z}(x_1)\mathcal{Z}(x_2)...\mathcal{Z}(x_n)} = \sum_\pi \exp[\frac{1}{2}\sum_{a,b=1}^{n} R_{\pi(a),\pi(b)} x_a x_b] \tag{74}$$

in the limit $n \to 0$. In the above expression, π denotes a permutation of the n replica indices, and the R_{ab} matrix is a one step RSB matrix [1] with elements $R_{ab} = r$ when a and b are in the same diagonal block of size m, $R_{ab} = 0$ when a and b are in different blocks, and $R_{aa} = (1-m)r$, enforcing the sum rule $\sum_{b=1}^{n} R_{ab} = 0$. By equivalent results we mean that one can compute quantities like the average probability of being in x,

$$\overline{P(x)} = \overline{\frac{\mathcal{Z}(x)}{\int dx' \mathcal{Z}(x')}} \equiv \int dx_2...dx_n \overline{\mathcal{Z}(x)\mathcal{Z}(x_2)...\mathcal{Z}(x_n)}\Big|_{n=0} \tag{75}$$

or the average of the product of the two probabilities to be in x and y,

$$\overline{P(x)P(y)} = \overline{\frac{\mathcal{Z}(x)\mathcal{Z}(y)}{(\int dx' \mathcal{Z}(x'))^2}} \equiv \int dx_3...dx_n \overline{\mathcal{Z}(x)\mathcal{Z}(y)\mathcal{Z}(x_3)...\mathcal{Z}(x_n)}\Big|_{n=0} \tag{76}$$

or generalizations thereof, by both methods (here the average is taken over the realization of the initial velocity profile). It has been shown in [20] (and we recall the main steps of the derivation in the appendix) that the velocity correlation function $\overline{v(x,t)v(y,t)}$ can be computed either directly from Eq. (72), or using the representation (74). The important points are the following:

– After a proper choice of length and velocity scales, the $\overline{v(x,t)v(y,t)}$ correlation function is indeed *identical* to the result obtained by Kida (see appendix), which is the consequence of the fact that both approaches actually rely on the universal structure of the extreme events which control the velocity field for large times.

– The present formalism allows us to extend Kida's results in several directions. For example, the full probability distribution function of $v(x) - v(y)$ has been computed in [20]. The problem of decaying Burgers turbulence in higher dimensions can also be addressed.

– The presence of shocks, which manifests itself as a $|x-y|$ singularity in $\overline{v(x,t)v(y,t)}$ at short distances, is intimately connected with the breaking of replica symmetry [20, 5]: for a replica symmetric matrix $R_{aa} = \tilde{R}$, $R_{a\neq b} = R_1$, $\overline{v(x,t)v(y,t)}$ is *regular* for $x \to y$. As discussed above, these shocks reflect, in the associated disordered problem, the existence of some *metastability* (see Fig. (2). From a technical point of view, it is interesting to see on this example how metastability is associated to RSB and, as emphasized in [5], to the existence of a short-distance singularity in the effective free-energy $\mathcal{V}^*$. Precisely the same behaviour is obtained via the Functional Renormalisation Group (FRG) [21]: a singularity appears in the renormalized correlation function of the effective free energy at scales larger than the 'Larkin length', which is the scale beyond which metastability effects become important. (However, the way to handle the shocks correctly within the FRG is still an open problem [5]).

6 Perspectives and other universality classes

As is the case for the central limit theorem, there are other universality classes, distinct from the Gaussian, when one relaxes the hypothesis of a finite variance or of independent variables (or both) [10]. This is also true for the statistics of extremes, and it is interesting to discuss how this might translate into a replica language. Two main directions can be thought of: independent variables with other types of distributions, or correlated variables.

Let us first consider the case where the energy levels E_i are still independent, but with a tail for large negative E decaying as a power-law, $|E|^{-1-\delta}$. In this case, the extreme values are distributed according to the so-called Fréchet distribution, which is different from the Gumbel distribution (for example, it decays asymptotically as a power-law with the same exponent δ). Rescaling E by $M^{1/\delta}$ to keep the gap between the ground state and first excited state finite as $M \to \infty$, one can calculate the quantities Y_k defined in (15). One finds, for M large but finite:

$$Y_k = 1 - \exp - \left(\frac{1}{T \log M}\right)^{\delta} \qquad (\delta < 1) \tag{77}$$

independently of k. (Similar results are obtained for $\delta > 1$). This is clearly different form Eq. (15). Notice that this case cannot be addressed within the replica method without some modifications since all the positive moments $\overline{\mathcal{Z}^n}$, $n > 0$ diverge !

Another universality class corresponds to E_i which are strictly bounded, i.e. $E_i = E_0 + \epsilon$, with $\epsilon \geq 0$. More precisely, the distribution of ϵ for $\epsilon \to 0$ is of the form $P(\epsilon) = \epsilon^{\delta}$ for ϵ small. The resulting distribution of extremes is then called the Weibull distribution. Rescaling the energies by a factor $M^{\delta+1}$, we find through a direct computation that Y_k is non trivial for all temperatures, i.e., the model is always in a low temperature phase. For instance in the case $\delta = 0$ one gets $Y_k(T \to \infty) \sim kT^{1-k}$, and $Y_k \sim 1 - CT$ for $T \to 0$ (C is Euler's constant). This is again clearly different from the 1 step RSB result (15). One might hope that such a situation will lead to a new type of RSB, but the situation seems more complicated. In the particular case $\delta = 0$, one finds:

$$\overline{\mathcal{Z}^n} = \sum_{i_1,\ldots i_n} \prod_i{}' \frac{T}{\sum_{a=1}^n \delta_{i,i_a}} \tag{78}$$

where the product is only over the sites i such that $\sum_{a=1}^{n} \delta_{i,i_a} > 0$. The entropy of the replica symmetric solution becomes negative below the temperature $T_c = 1/e$. Assuming a one step RSB saddle point for $T < T_c$ leads to a constant average energy, equal to $1/e$; however, the true ground state energy can be calculated directly and is equal to 1. Again, this result is not of the replica type. In this case however, a sensible replica calculation can be undertaken since all the $\overline{\mathcal{Z}^n}$, $n > 0$ are convergent. The problem now is that the free-energy cannot be calculated by a *saddle point* method: replica fluctuations are always important.

In any case, from the point of view of Burgers' turbulence, it is interesting to notice that initial conditions for the velocity field which do not belong to the exponential universality class considered by Kida will lead to rather different flow structures at long times, even within the class of $E(x)$ functions with local correlations.

Let us now turn to the case where the E_i are Gaussian but long-range correlated; for example the case where E depends on a d-dimensional space variable $\vec{x}$, and such that:

$$\overline{\tilde{E}(\vec{q})\tilde{E}(\vec{q}')} = \frac{\delta(\vec{q}+\vec{q}')}{q^{2-\eta}} \tag{79}$$

leading to

$$\overline{(E(\vec{x}) - E(\vec{y}))^2} \propto |x-y|^{\max(0,2-d-\eta)} \tag{80}$$

The case $\eta = 0$, $d = 1$ corresponds to a random walk for $E(x)$, which has been studied in detail both in the context of Burgers' turbulence [8], and also as a partly solvable spatial REM [14, 22, 23]. The general η, d case has not been solved yet. It has been studied by the Gaussian variational replica formalism of [7], which shows [15] that the case $\eta < 2 - d$ (corresponding to a growing correlation function (80)) requires 'continuous' RSB, while the case $\eta > 2 - d$ only requires a 'one step' breaking. Independent variables correspond to $\eta = 2$, i.e. a white spectrum for $E(\vec{q})$. We conjecture here that the case $\eta > 2 - d$ belongs to the same (one step RSB) universality class as the REM ($\eta = 2$). It is actually not difficult to show directly that the quantities:

$$c_n = \frac{\overline{\mathcal{Z}^n} - \overline{\mathcal{Z}}^n}{\overline{\mathcal{Z}}^n} \qquad n = 2, 3, .. \tag{81}$$

diverge with the system size below a certain n dependent critical temperature which is independent of η for $\eta > 2 - d$, and identical to those found in the REM. This suggests that the one step solution indeed remains *exact* for all $\eta > 2 - D$. Preliminary numerical simulations [24] seem to confirm this. This points towards a rather natural result, namely the fact that *weak enough correlations* (measured here by $2 - \eta$) between the random variables *does not change the universality class for the extreme value statistics*. This is actually a Theorem for the case $d = 1$: for all $\eta > 1$, the extreme value statistics is indeed of the Gumbel type [25], while some corrections appear in the marginal case $\eta = 1$. Interestingly, a related conjecture was proposed recently for the $d = 2$ problem with $\eta = 0$, which corresponds to the localization of electrons in a random magnetic field [13]. In this two dimensional case, the choice $\eta = 2 - d = 0$ corresponds to a marginal logarithmic growth of the correlations.

Returning to the one dimensional case, the situation changes drastically when $\eta < 1$, which corresponds to a typically 'growing' profile $E(x)$. The ratios c_n diverge with the system size for all temperatures, suggesting indeed a change of universality class. The only known possibility at present is then to describe the system within a 'continuous' RSB, which can be interpreted as a recursive tree-like construction of the low-lying energy state. In particular, the correlation of the low-lying states have a well known ultrametric structure. How well this ultrametric structure (known to be exact for the case where the dimension of $\vec{x}$ is infinite) reproduces the distribution and correlations of the low lying states in finite dimension is an open problem [5]. We leave this problem for further studies [24].

In summary, we have argued for the one dimensional problem that the one step replica symmetry breaking scheme encodes exactly the results on the statistics of extremes for variables which are:

i) not too correlated (i.e. when the above exponent η is larger than $2-d$)

ii) distributed asymptotically as generalized exponentials (i.e. as $\exp -|E|^{\delta}$).

The case of long range correlations may correspond, in some limit, to the 'continuous' RSB scheme. However, the precise link between the two is not clear to us at present, and we think that it is an important path to explore further. It would also be interesting to think of the spin-glass problem from the point of view of the classification of very low energy states and excitations, which could perhaps provide a natural link between replicas and 'droplet-like' descriptions [26].

ACKNOWLEDGMENTS We thank A. Baldassarri, R. Cont and V. Dotsenko for many interesting discussions. MM thanks the SPhT in the CEA Saclay for its hospitality.

Appendix

In this appendix we explain briefly how the replica method and the direct probabilistic analysis lead to the same result for the two point correlation function in decaying Burgers turbulence at large times. We are interested in the case where $E(x)$ has local correlations (see section 5.3), in which case the result of Kida reads:

$$\overline{v(x)v(x+r)} = v_{\max}^2 H\left(\frac{r}{\ell(t)}\right) \tag{82}$$

where $\ell(t)$ is the mean distance between two shocks, $v_{\max} = \ell(t)/t$, and

$$H(x) \equiv \frac{1}{\sqrt{2\pi}} \frac{d}{dx} x \int_0^\infty \frac{dy}{\phi(x+y)+\phi(x-y)} \tag{83}$$

and ϕ is an error function: $\phi(x) = \int_0^\infty dz \exp(-z^2 + \sqrt{\pi/2}\, xz)$.

We shall sketch how these results can be obtained from the replica representation (74). The computations are lengthy and already contained in some previous papers. Here we just want to help the interested reader to find his way in the litterature in order to obtain the result. One starts from the replicated partition function (74):

$$\overline{\mathcal{Z}(x_1)\mathcal{Z}(x_2)...\mathcal{Z}(x_n)} = \sum_{\pi} \exp[\frac{1}{2} \sum_{a,b=1}^{n} R_{\pi(a),\pi(b)} x_a x_b] \ , \tag{84}$$

where the R_{ab} matrix is a one step RSB matrix [1] with elements $R_{ab} = r$ when a and b are in the same diagonal block of size m, $R_{ab} = 0$ when a and b are in different blocks, and $R_{aa} = (1-m)r$, enforcing the sum rule $\sum_{b=1}^n R_{ab} = 0$. The first step, derived in the appendix D of [20], deduces from (84) the correlation between the powers $n/2$ of the partition function:

$$\overline{Z(x,t)^{n/2} Z(y,t)^{n/2}} = \frac{2^n}{B(-n/2,-n/2)} \int_0^\infty \frac{d\mu}{\mu} \mu^{-n/2} \left[\frac{m\sqrt{r_1}}{2\pi} \int dz\, (e_x + \mu e_y)^m\right]^{n/m} \ . \tag{85}$$

[5] Note that the average ground state energy predicted by the Gaussian *variationnal* replica theory does not lead back to the exact result [23] in the soluble random walk case $\eta = 0$.

where we have defined

$$e_x \equiv e^{-mr(z-x)^2/2} \quad , \quad e_y = e^{-mr(z-y)^2/2} \tag{86}$$

Using the general link $v = -2\nu \frac{\partial \log \mathcal{Z}}{\partial x}$, one gets:

$$\overline{v(x)v(y)} = \lim_{n\to 0} \frac{16\nu^2}{n^2} \frac{\partial^2}{\partial x \partial y} \overline{Z(x,t)^{n/2} Z(y,t)^{n/2}} = (2\nu m r)^2 \left(g_{11} + g_{12}\right) \tag{87}$$

where we have defined (the notations are those of [20] -Appendix B):

$$g_{11} = (1-m)\int_0^\infty d\mu \, \frac{\int dz \, (e_x + \mu e_y)^{m-2} \, (x-z) e_x (y-z) e_y}{\int dz \, (e_x + \mu e_y)^m} \tag{88}$$

and:

$$g_{12} = m \int_0^\infty d\mu \, \frac{\left(\int dz \, (e_x + \mu e_y)^{m-1} \, (x-z) e_x\right) \, \left(\int dz \, (e_x + \mu e_y)^{m-1} \, (y-z) e_y\right)}{\int dz \, (e_x + \mu e_y)^m} \; . \tag{89}$$

This expression could also be derived directly without replicas from the infinite sum (72), with the identification $m = \mu$, $r = \kappa/(T\mu)$ as can be seen from formulas (B10,B11) of [20] (where the number $1/(mr)$ was called δ).

The whole problem is now to evaluate this expression in the limit of large Reynolds, which means small μ or low T. In this regime, using the fact that m scales linearly with T and r scales as $1/T^2$, it has been shown in the appendix B of [20] that expression (87) reduces to:

$$\begin{aligned} \overline{v(x)v(y)} \;&=\; \frac{\kappa T}{\mu} \left(\sqrt{\frac{2}{\pi}} \int_0^\infty dh \frac{e^{-h^2/2} \, [h^2 - d^2/4]}{e^{d^2/8} \left[e^{-hd/2} \mathcal{M}_0(h - \frac{d}{2}) + e^{hd/2} \mathcal{M}_0(-h - \frac{d}{2}) \right]} \right. \\ &\left. -\; \frac{d}{\pi} \int_0^\infty dh \frac{e^{-h^2}}{e^{d^2/4} \left[e^{-hd/2} \mathcal{M}_0(h - \frac{d}{2}) + e^{hd/2} \mathcal{M}_0(-h - \frac{d}{2}) \right]^2} \right) \end{aligned} \tag{90}$$

where

$$\mathcal{M}_0(x) = \int_x^\infty \frac{dz}{\sqrt{2\pi}} e^{-z^2/2} \quad ; \quad d = \frac{|x-y|}{\sqrt{T/(\mu\kappa)}} \tag{91}$$

So the natural length scale appearing in this solution is $\ell = \sqrt{T/(\mu\kappa)}$. Using (73,69), one sees easily that it precisely scales at large times as the average distance between shocks of Kida's analysis. In terms of reduced lengths, one can check that the two distributions (90) and (83) are actually identical.

References

[1] For a review, see: M. Mézard, G. Parisi, M.A. Virasoro, "Spin Glass Theory and Beyond", (World Scientific, Singapore 1987)

[2] E. J. Gumbel, *Statistics of Extremes*, Columbia University Press, (1958)

[3] J. Galambos, *The asymptotic Theory of Extreme Order Statistics*, R.E. Krieger Publishing Co., Malabar, Florida (1987).

[4] For the use of extreme value statistics for energy barriers in disordered systems, see: R. Rammal, J. Physique **46** 1837 (1985), [5], and V. Vinokur, M. Cristina Marchetti, L. W. Chen, Phys. Rev. Lett. **77** 1845 (1996).

[5] L. Balents, J.P. Bouchaud, M. Mézard, J. Physique I **6** 1007 (1996).

[6] B. Derrida, Phys. Rev. **B 24**, 2613 (1981); D.J. Gross, M. Mézard, Nucl. Phys. **B240** 431 (1984).

[7] M. Mézard, G. Parisi, J. Physique I **1** 809 (1991); J.Phys. **A23** L1229 (1990)

[8] J. M. Burgers, 'The Non-Linear Diffusion Equation', D. Reidel Pub. Co. (1974)

[9] S. Kida, J. Fluid. Mech. **93** 337 (1979).

[10] For a review, see: J.P. Bouchaud, A. Georges, Phys. Rep. **195** 127 (1990), Chapter I.

[11] B. Derrida, *Non-Self averaging effects in sum of random variables*, in 'On Three Levels', Edited by Fannes et al, page 125 Plenum Press, NY, 1994

[12] M. Mézard, G. Parisi, N. Sourlas, G. Toulouse and M. A. Virasoro, *Phys. Rev. Lett.* **52** 1156 (1984), *J. Physique* **45** 843 (1984).

[13] for a recent work related to the present framework, see: C. Chamon, C. Mudry, X.G. Wen, Phys. Rev. Lett. **77** 4194 (1996); H. E. Castillo, C. Chamon, E. Fradkin, P. M. Goldbart, C. Mudry, preprint cond-mat 9706084.

[14] U. Schultz, J. Villain, E. Brezin, H. Orland, J. Stat. Phys. **51** 1 (1988); J. Villain, B. Semeria, F. Lançon, L. Billard, J. Phys. **C 16** 6153 (1983).

[15] M. Mézard, G. Parisi, J. Physique I **2** 2231 (1992)

[16] C. Monthus, J.P. Bouchaud, J. Phys. A 29 3847 (1996).

[17] A. Engel, Nucl. Phys. **B 410** 617 (1993).

[18] C. Caroli, P. Nozières, in *Physics of Sliding Friction* B.N.J. Persson and E. Tosatti edts; NATO ASI Series, Applied Sciences, 311 (Kluwer Academic).

[19] S.N. Gurbatov, S.I. Simdyankin, E. Aurell, U. Frisch, G. Tóth, *On the decay of Burgers turbulence*, to appear in J. Fluid Mech. (1997).

[20] J.P. Bouchaud, M. Mézard, G. Parisi, Phys. Rev. **E52** 3656 (1995).

[21] D.S. Fisher, Phys. Rev. Lett. **56** (1986) 1964; L. Balents, D. S. Fisher, Phys. Rev. **B 48** (1993) 5949

[22] C. Monthus, A. Comtet, J. Physique I **4** 635 (1994)

[23] K. Broderix, R. Kree, *Thermal equilibrium with the Wiener potential: testing the replica variationnal approximation*, cond-mat 9507099.

[24] A. Baldassarri, J.P. Bouchaud, M. Mézard, in preparation.

[25] see Theorem 3.8.2, [3].

[26] see e.g. D.S. Fisher, D.A. Huse, Phys. Rev. Lett **56**, 1601 (1986); Phys. Rev. **B 38**, 373 (1988).